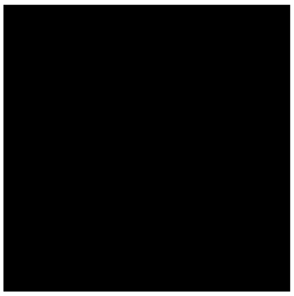
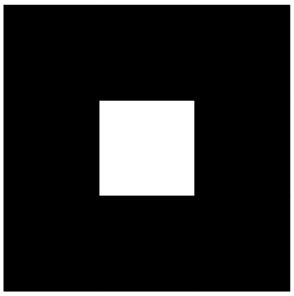
MAT 141
Problem Set #5

due in recitation on October 7 or 8, 2004

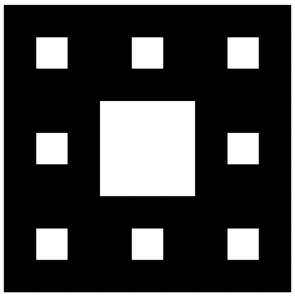
1. Apostol, section 1.7 # 1, 2
2. Apostol, section 1.11 # 1, 2, 6
3. Prove that the union of finitely many rectangles is measurable.
4. We give an inductive definition of a family of sets, S_n. S_0 is the unit square in the plane; that is, $S_0 = \{(x, y) \mid 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$. To construct S_{k+1} from S_k, we do the following. Subdivide the unit squre into 9 smaller squares in a tic-tac-toe board pattern. Leave the center square empty and fill each of the remaining squares with a copy of S_k, which has been shrunk by a factor of 1/3. The first couple of S_k's are pictured below.



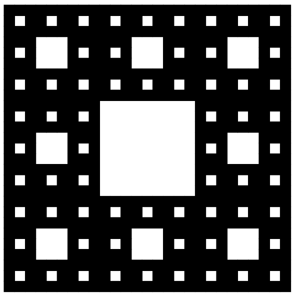
S_0



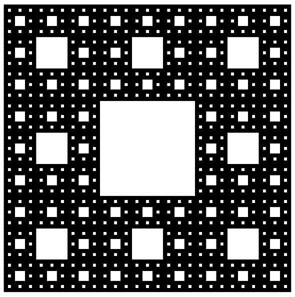
S_1



S_2



S_3



S_4

Let $S = \bigcap S_k$; that is, S consists of those points in the plane that are contained in every S_k.

(a) Prove that each S_k is the union of finitely many squares, and hence, measurable.

(b) Derive a formula for the area of S_k and prove that it is correct.

(c) Use the fact that S is contained in each of the S_k's to prove that the infimum of the set

$$\{a(T) \mid T \text{ is measurable and } S \subseteq T\}$$

is zero.

(d) Prove that S is measurable. Compute the area of S.

S, is called the "Sierpinski carpet" and is an example of a fractal, which is short for "fractional dimension". While we usually think of dimension as being an integer (a line is one dimensional, a square is two dimensional, and a cube is three dimensional), the dimension of S is $\log_3 8 \approx 1.8928$.