WSE 187: Complex numbers and the Mandelbrot set 3/03/2003

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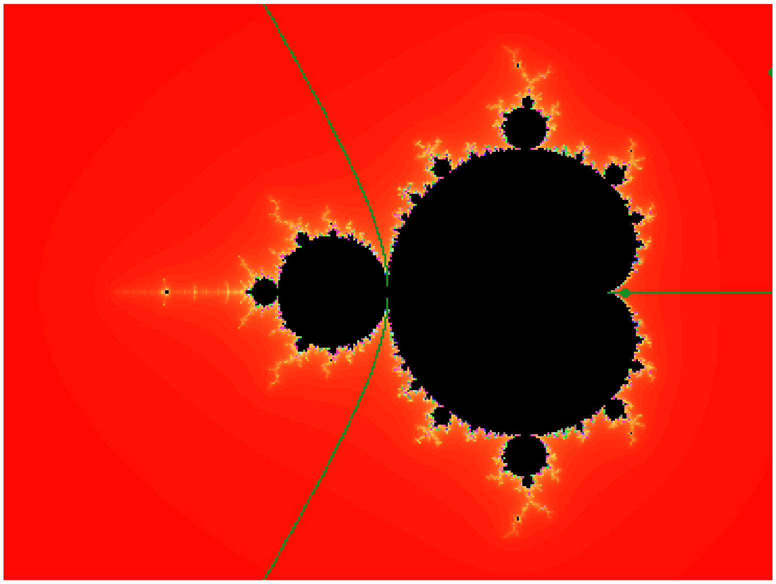

Figure 1: The M-Set

Complex numbers

Complex numbers are written $a+ib$ or sometimes $a+\sqrt{-1}b$, where $a, b \in \mathbb{R}$. They add and multiply according to the rules:

$$(a+ib)+(c+id)=a+c+i(b+d), \quad (a+ib)(c+id)=ac-bd+i(bc+ad).$$

Examples: $i^2=-1, \ (1+3i)(1-3i)=1^2+3i-3i+(3i)^2=1-(-9)=10.$

There is a trick that helps us **divide** one complex number by another: Note that

$$(a+ib)(a-ib)=a^2+b^2.$$

This is a real positive number (unless $a+ib=0$.) So we can covert any expression $\frac{a+ib}{c+id}$ into the form $x+iy$ by **rationalizing the denominator**, ie multiplying top and bottom by $c-id$.

Example

$$\frac{1}{1+2i}=\frac{1}{1+2i}\frac{1-2i}{1-2i}=\frac{1-2i}{(1+2i)(1-2i)}=\frac{1-2i}{1+2^2}=\frac{1}{5}-\frac{1}{2}i.$$

We will concentrate here on addition and multiplication.

Real numbers can be pictured on a line. Complex numbers can be pictured on a plane. We plot $a+ib$ as the point (a,b); we can also look at the corresponding **vector** from the origin $(0,0)$ to (a,b). Addition and multiplication have very nice pictorial interpretations.

Parallelogram rule for addition The points $(0,0)$, (a,b), (c,d) and the sum $(a+c, b+d)$ form the vertices of a Parallelogram.

Multiplication in polar coordinates Instead of describing the point $z = a+ib$ on the plane by rectangular coordinates we use polar coordinates; $r = \sqrt{a^2+b^2}$ and θ where $\tan\theta = b/a$. Thus the point $a+ib$ can be written as

$$a + ib = r\cos\theta + ir\sin\theta.$$

We define $re^{i\theta}$ to be this number

$$re^{i\theta} := r\cos\theta + ir\sin\theta.$$

$r = \sqrt{a^2+b^2}$ is called the **absolute value** of $a+ib$ and is written $|a+ib|$. **Example** $i = e^{i\pi/2}$; $\;-1 = e^{i\pi}$; $\;1+i = \sqrt{2}e^{i\pi/4}$ $\;|i+i| = \sqrt{2}$.

The next calculations show that the multiplication rules for a number in this form are just what you would expect from the exponential notation. ie we would expect

$$re^{i\theta}\cdot se^{i\phi} = rse^{i(\theta+\phi)} :$$

"distances multiply and angles add".

Here is the general calculation:

$$\begin{aligned} re^{i\theta}se^{i\phi} &= (r\cos\theta + ir\sin\theta)(s\cos\phi + is\sin\phi) \\ &= rs(\cos\theta\cos\phi + i^2\sin\theta\sin\phi) + rs(i\sin\theta\cos\phi + i\cos\theta\sin\phi) \\ &= rs\Big(\cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + i^2\cos\theta\sin\phi)\Big) \\ &= rs(\cos(\theta+\phi) + i\sin(\theta+\phi)) \\ &= rse^{i(\theta+\phi)}. \end{aligned}$$

How to picture maps or functions $f : \mathbb{C} \to \mathbb{C}$

You can't draw a graph (it would have to be in 4-dimensions!), so we draw pictures in the plane explaining where the different points go.

Addition is **translation**; Multiplication by $e^{i\theta}$ is **rotation**; The map $z \mapsto f(z) = z + c$; multiplication by r stretches (or shrinks) by the factor r.

Dynamical systems

One way of trying to understand a map $f : X \to X$ is to repeat it over and over again and see what happens to the points z. The set of points

$$z, f(z), f(f(z)), f(f(f(z))), \ldots$$

is called the **orbit** of z. We also write $z \mapsto f(z) \mapsto f(f(z)) \mapsto \ldots$ Understanding the map f as a dynamical system means understanding its orbits.

We will look at **quadratic maps** of the form $f_c(z) = z^2 + c$, for c a fixed complex number. These maps are 2 to 1; i.e. every point w (except for $w = c$) has **two preimages**; i.e. for each w there are two solutions to the equation $f(z) = w$:

$$f(z) = w \Leftrightarrow z^2 + c = w \Leftrightarrow z^2 = (w - c) \Leftrightarrow z = \pm\sqrt{w - c}.$$

Example 1: the map $f(z) = z^2$

$f(1) = 1^2 = 1$: so 1 is a **fixed point**

$-1 \mapsto (-1)^2 = 1 \mapsto 1 \mapsto 1 \ldots$: so the orbit of -1 is eventually fixed.

$i \mapsto i^2 = -1 \mapsto 1 \mapsto 1 \ldots$; also $-i$ eventually arrives at 1.

Angle doubling for points on the unit circle: $z = e^{i\theta}$ goes to $z^2 = e^{2i\theta}$, so the angle θ doubles to 2θ.

$$e^{\pi i/3} \mapsto e^{2\pi i/3} \mapsto e^{4\pi i/3} \mapsto e^{8\pi i/3} = e^{2\pi i/3} \mapsto e^{4\pi i/3} \ldots$$

we are again in a repeating pattern: we say that this orbit is **eventually periodic** (with period 2.)

Points on the circle remain on it. What about points inside? They converge to zero, while points outside the unit circle diverge to ∞.

$$2 \mapsto 4 \mapsto 16 \mapsto 256 \mapsto \ldots, \quad \text{goes to } \infty;$$

$$2i \mapsto -4 \mapsto 16 \mapsto 256 \mapsto \ldots \quad \text{goes to } \infty;$$

$$i/2 \mapsto -1/4 \mapsto 1/16 \mapsto 1/256 \ldots \quad \text{goes to } 0.$$

The **filled Julia set** (sometimes referred to as the Julia set) for this map $z \mapsto z^2$ is the set of points with bounded orbits under this map. Hence it is the disc of radius 1.

In Figure 2 points that are colored black have orbits that stay bounded (in other words the black circle represents the Julia set). The points whose orbits escape to ∞ are colored red. The interior of the unit circle contains all the points whose orbits converge to 0 and its boundary consists of all points whose orbits remain on the unit circle.

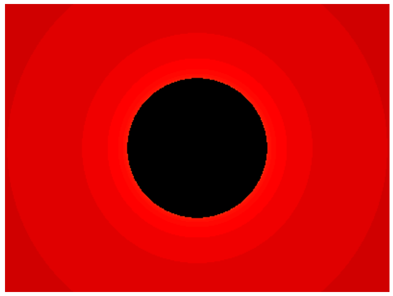

Figure 2: The filled Julia set for z^2

Example 2: the map $z \mapsto z^2 - 1$

This function has a periodic orbit of period 2: $0 \mapsto -1 \mapsto 1^2 - 1 = 0 \mapsto 1 \mapsto 0 \ldots$.

Figure 3 represents the filled Julia set for $z^2 - 1$. Points with bounded orbits are colored black and points whose orbits tend to ∞ are marked red. In fact all the points in the interior of the black region have orbits attracted to the cycle $0, -1, 0, -1 \ldots$ Such a cycle is called an attracting cycle and the interior of the Julia set is the basin of attraction for that cycle. Again, the boundary of the black region, which is a very complicated geometric object, called fractal, consists of points whose orbits remain on this boundary. Observe that in both cases the filled Julia sets were connected, and the point 0 was colored black.

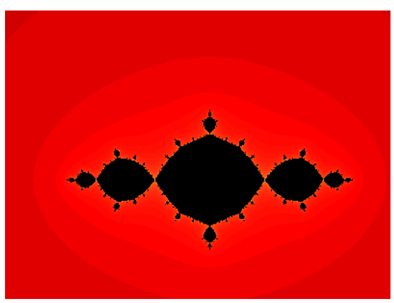

Figure 3: The filled Julia set for $z^2 - 1$

Example 3: the map $z \mapsto z^2 + i$

This map also has a periodic orbit of period 2: $-i \mapsto -1 + i \mapsto -i \mapsto -1 + i \ldots$, but it is very different from example 2. because the **filled Julia set** coincides with the **Julia set** and therefore does not have any interior. The Julia set is still connected, that is it contains only one piece. The yellow color in Figure 4 surrounds the Julia set and contains all the black points whose orbits are bounded. The shape, which is again a fractal is known as the **dendroid**. It is a connected set with infinitely many branches and the point 0 lies on it.

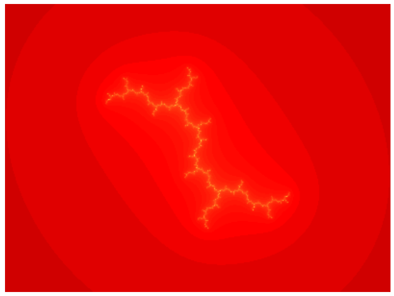

Figure 4: The filled Julia set for $z^2 + i$

In all of the three examples above, the point 0 was part of the filled Julia set, thus its orbit remained bounded under the corresponding quadratic function. Now we will consider a different example.

Example 4: the map $z \mapsto z^2 + 0.3$

It can be verified that the orbit of 0 in this case gets larger and larger $0 \mapsto 0.3 \mapsto 0.39 \mapsto 0.5421 \mapsto \ldots$ and therefore converges to ∞.

The corresponding Julia set is a disconnected set of uncountably many points called **fractal dust**. These points are hidden behind the yellow color in Figure 5 and they form a disconnected set. Since the orbit of 0 tends to ∞, it is colored red and does not belong to the fractal dust.

Figure 5: The filled Julia set for $z^2 + 0.3$

Reminder Each point c in M determines a dynamical system (or map) $f_c : z \mapsto z^2+c$. The dynamics of f_c are pictured in the dynamic plane and the Julia set J_c is part of it.

Question1: Consider the case $c = +i$: Calculate the orbit of $-i$. What is the orbit of 0?

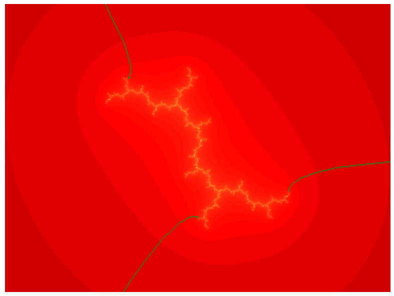

Figure 6: $\frac{1}{3}, \frac{2}{3}$ and 0 External rays for $z^2 + i$

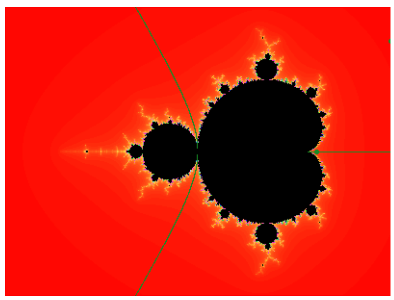

Figure 7: $\frac{1}{3}, \frac{2}{3}$ and 0 External rays for the Mandelbrot set

Question 2: Investigate the doubling map on the circl e: $t \mapsto 2t$ where $t \in [0, 1]$ (so the point $t = 0$ is the same as the point $t = 1$.) We can think of t as a real number that we reduce modulo 1. So $t = 0$ is a fixed point.
(i) Find the orbits of the points $1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9 \ldots$
(ii) Try other points (such as $2/3, 3/5, 3/7 \ldots$.) Also try $1/15, 1/31$ (in general $1/2^n - 1$.)
(iii) What periodic orbits do you find? What patterns do you see? We will discuss this next time.

Differentiating complex functions

Functions of the complex variable z can be differentiated just as though z were real. So:

the function $f(z) = z^2 + c$ has derivative $f'(z) = 2z$.

The question is: what does this mean? It's best to interpret this via the **linear approximation formula** ie for points z close to v the map $z \mapsto f(z)$ looks very similar to the linear map $z \mapsto f(v) + f'(v)(z - v)$. This map multiplies the small distance $z - v$ by $f'(v)$ and then translates by $f(v)$. So

$$f(z) - f(v) \approx f'(v)(z - v).$$

So if $|f'(v)| < 1$, the images $f(z), f(v)$ of z, v under f are closer than they were before, while if $|f'(v)| > 1$ the images are further apart.

The derivative and periodic orbits

Suppose that $f(v) = v$, ie v is a fixed point. Then if $|f'(v)| < 1$ nearby points get attracted to v while if $|f'(v)| > 1$ nearby points are repelled. Thus a fixed point can be **attracting** or **repelling**. The case $|f'(v)| = 1$ is specially interesting because it is a borderline case. Look at the pictures to see what can happen here!

We can extend this idea to periodic orbits. Suppose that $v_0 \mapsto v_1 \mapsto \ldots \mapsto v_q = v_0$ is a periodic orbit of period q under f and let $F = f \circ f \circ \ldots \circ f$ be the q-fold composite of f. Then $F(v_0) = v_0$. By the chain rule

$$F'(v_0) = f'(v_0) \cdot f'(v_1) \cdots f'(v_{q-1}),$$

the product of the derivatives along the orbit. If $|F'(v_0)| < 1$ then nearby points are attacted to the periodic orbits while if $|F'(v_1)| > 1$ they are repelled.

Periodic orbits for the quadratic family

The map $f_c(z) = z^2 + c$ has derivative $f'(z) = 2z$. Therefore it has exactly one critical point, namely $z = 0$. It turns out that the orbit of this point is very special: if there is an **attracting periodic orbit** then the orbit of 0 must converge to it. If this happens the orbit of 0 is bounded and hence c is in the Mandelbrot set. In fact most of the **the Mandelbrot set M consists of the c values for which f_c has an attracting periodic orbit of some period q**. However there are cases where the orbit of 0 may stay bounded without being attracted to an attracting cycle.

Since $f_c(0) = c$, c is the *critical value*. Since $0 \mapsto c \mapsto f_c(c) \ldots$ the orbit of c also converges to the attracting periodic orbit (if there is one.)

Every quadratic map has lots of periodic orbits. We will look into the fixed points and periodic points of period two. The fixed points are just solutions of the equation $z^2 + c = z$, so there are two; They are given by the formula

$$p_{\pm} = \frac{1}{2}\left(1 \pm \sqrt{(1-4c)}\right).$$

Periodic orbits of period 2 are solutions of the equation

$$f(f(z)) = f(z^2 + c) = (z^2 + c)^2 + c = z$$

which has degree 4 and so has 4 solutions. The solutions are given by the fixed points of the first iterate, namely $p\pm$ and by two additional points:

$$q_{+,-} = \frac{1}{2}\left(-1 \pm \sqrt{(-4c-3)}\right)$$

The periodic points of period 3 (including the fixed points) satisfy the equation:

$$f(f(f(z))) = f(f(z^2 + c)) = (((z^2 + c)^2 + c)^2 + c = z.$$

As you can see the degree of this polynomial is eight and there are eight solutions (two fixed points and two cycles of period three). The degree of the polynomial representing the n-th iterate of f is 2^n and the solutions to $f^n(z) = z$ provide all possible orbits of period n (not necessarily prime). An important fact that we mentioned earlier is that at most one of these infinitely many periodic orbits is attracting. However there may be no attracting cycle.

The attracting periodic orbits for the maps f_c get extra structure from the **external rays** – using these one can imagine that the points of the orbit lie on the circle. Hence they have different types one for each fraction p/q: see Question 3 below.

Each of the bulbs on the main cardioid of the Mandelbrot set M (called also hyperbolic components) corresponds to a different rational number p/q. You can often read off p/q from the picture of the bulb or from the Julia sets coming from it, or from the external rays landing at their root points.

Why the main bulb in the Mandelbrot set is a cardioid

The main bulb of the Mandelbrot set consists of c values for which the map f_c has an attracting fixed point. It turns out that this point is

$$p_-(c) = \frac{1}{2}\left(1 - \sqrt{(1 - 4c)}\right).$$

Therefore for each c in the main bulb $|f_c'(p_-(c)| = 2|p_-(c)| < 1$, or

$$|p_-(c)| < \frac{1}{2}.$$

On the boundary of this bulb the inequality turns into the equality $|p_-(c)| = \frac{1}{2}$ or $p_-(c) = \frac{1}{2}e^{2\pi it}$ for $t \in [0, 1]$. In order to solve for c we remember that

$$p_-(c)^2 + c = p_-(c)$$

and therefore the boundary of the cardioid is:

$$c(t) = p_-(c) - p_-(c)^2 = \frac{1}{2}e^{2\pi it} - \frac{1}{4}e^{4\pi it}.$$

As t varies in $t \in [0, 2\pi)$, this gives the equation of a cardioid. Note that the derivative of f_c at $p_-(c)$ is precisely $2p_-(c) = e^{2\pi it}$, a rotation through angle $2\pi t$. If $t = p/q$ the point $c(t)$ is the attaching/root point of the p/q bulb. This is why we can often see the number p/q from the picture of the p/q bulb.

Question 3: Structure of periodic orbits of period q under rotation.

We can investigate this by looking at rotations of the circle by angle p/q. Think of the circle as the real numbers modulo 1 so that rotation is just addition. Every orbit has period q:

$$t \mapsto t + \frac{p}{q} \mapsto t + \frac{2p}{q} \mapsto t + \frac{3p}{q} \mapsto \ldots \mapsto t + \frac{qp}{q} t + p \equiv t.$$

The order of the image points on the circle depends on p: Example: If $p/q = 1/3$ then $0 \mapsto 1/3 \mapsto 2/3 \mapsto 0$, while if $p/q = 2/3$ then $0 \mapsto 2/3 \mapsto 1/3 \mapsto 0$. Experiment with the orbits of rotation by $p/5$, $p/7$ etc. Draw pictures on the unit circle connecting corresponding points under the different rotations.

Binary expansions, itineraries of fractions and why it matters.

Every number in the interval $[0,1)$ can be written in the form

$$p/q = 0.s_1 s_2 ... s_n ... = \frac{s_1}{2} + \frac{s_2}{2^2} + \ldots \frac{s_n}{2^{n+1}} + \ldots$$

where $s_n = 0$ or $s_n = 1$, $n = 1, 2, \ldots$.

For example 0.01 is the binary expansion of $\frac{1}{4}$ because $\frac{0}{2} + \frac{1}{2^2} = \frac{1}{4}$. In the same way the binary expansion of $\frac{3}{4}$ is 0.11 because $\frac{1}{2} + \frac{1}{2^2} = \frac{3}{4}$. On the other hand $0.\overline{101}$ represents a self repeating infinite binary number $0.101101101\ldots101\ldots$ whose sum is an infinite geometric series

$$\frac{1}{2} + \frac{0}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{0}{2^5} + \frac{1}{2^6} + \ldots,$$

which can simply be computed as:

$$\left(\frac{1}{2} + \frac{0}{2^2} + \frac{1}{2^3}\right)\left(1 + \frac{1}{2^3} + \frac{1}{2^6} + \ldots\right) = \frac{5}{8}\frac{1}{1 - \frac{1}{2^3}} = \frac{5}{7}$$

Itineraries

Let's divide the interval $[0,1)$ into two subintervals $I_0 = [0, \frac{1}{2})$ and $I_1 = [1/2, 1)$. The **itinerary** of the point $\frac{p}{q} \in [0,1)$ is an infinite sequence $s_0 s_1 ... s_n \ldots$, where $s_n = 0$ or $s_n = 1$, as before. Here s_k is 0 if the k^{th} iterate of p/q under the doubling function is in I_0 and, alternatevely, s_k is 1 if the k^{th} iterate of p/q under the doubling function is in I_1. Note that here 0 lies in I_0 and $1/2$ lies in I_1.

For example the itinerary of $x_0 = 0$ is $0000\ldots0\ldots$, because all of its iterates being equal to 0 lie in I_0. The orbit of $\frac{1}{3}$ under the doubling function is $\frac{1}{3} \mapsto \frac{2}{3} \mapsto \frac{1}{3} \mapsto \frac{2}{3} \ldots$, which is an orbit of period 2 with itinerary $0101010101\ldots$, simply written as $\overline{01}$. The orbit of $\frac{2}{5}$ under the doubling function is $\frac{2}{5} \mapsto \frac{4}{5} \mapsto \frac{3}{5} \mapsto \frac{1}{5} \mapsto \frac{2}{5} \ldots$. This is a cycle of period 4 with itinerary of $01100110\ldots = \overline{0110}$ It is not difficult to check that the binary number $0.\overline{0110} = \frac{2}{5}$.

In general the following rules hold:

(1) If the itinerary of p/q is $s_1s_2\ldots s_n\ldots$ then the corresponding binary expansion $0.s_1s_2\ldots s_n\ldots$ equals $\frac{p}{q}$ i.e. $0.s_1s_2\ldots s_n\ldots = p/q$.

(2) If the itinerary of p/q is $s_1s_2\ldots s_n\ldots$ then the itinerary of its image under the doubling function is $s_2\ldots s_n\ldots$, which is equivalent to deleting the first integer in the itinerary.

(3) To find the n^{th} iterate of p/q with itinerary $s_1s_2\ldots s_ns_{n+1}s_{n+2}\ldots$ we can simply find the number with binary expansion $0.s_{n+1}s_{n+2}\ldots$. For example take the number $\frac{2}{5}$, whose itinerary is $01100110\ldots = \overline{0110}$. Its third iterate $\frac{3}{5}$ has itinerary $001100110011\ldots = \overline{0011}$ and of course binary expansion $0.\overline{0011}$. Check this.

(4) A number $\frac{p}{q}$ lies on a cycle of period n under the doubling function if its itinerary can be written as the same block of length n containing 0's and 1's repeated infinitely many times. For example $\frac{1}{3}$ lies on a cycle of period 2 and its itinerary $0101010101\ldots = \overline{01}$ consists of the block 01 repeated infinitely many times.

(5) If a number $\frac{p}{q} \in [0,1)$ has period n under doubling it can be written in the form $\frac{p}{2^n-1}$. You should try to prove this fact.

(6) Check the following formula for the fraction p/q with itinerary $s_1s_2\ldots s_k$: $q = 2^k - 1$ and p is the integer with binary representation $s_1s_2\ldots s_k$: thus

$$p = s_1 + 2s_2 + 2^2s_3 + \ldots + 2^{k-1}s_k.$$

Question 4:

1. Find the rational numbers whose binary expansions are $0.\overline{010}$, $0.\overline{01111}$.
2. Find the itineraries and the binary expansions of the following numbers: $\frac{1}{3}, \frac{1}{4}, \frac{15}{31}$.

Fact 1.: Prove that all cycles of period $3, 4$ and 5 under the doubling function can be written as rational numbers in the form $\frac{p}{2^3-1}, \frac{p}{2^4-1}, \frac{p}{2^5-1}$, where p is an integer less than 2^3-1, 2^4-1 or 2^5-1, respectively.

Note. If a rational number $\frac{p}{q} \in [0,1)$ has an itinerary of period n, then it lies on a cycle of period n. This cycle is not necessarily prime. In other words, if the number n is not prime the cycle may begin repeating itself before the nth iterate. A cycle of length n is called prime if this does not happen. (e.g $\frac{1}{3}$ lieas on cycles of period $2, 4..$ but its prime cycle has length 2.

The $\frac{p}{q}$ bulbs

The boundary of the cardioid of the Mandelbrot set is defined by the function

$$c_t = c(t) = \frac{1}{2}e^{2\pi it} - \frac{1}{4}e^{4\pi it}.$$

There are infinitely many bulbs attached to the main cardioid. The point of attachment is called a root point of the bulb. If a bulb has a root point $c_{\frac{p}{q}}$ then it is called the $\frac{p}{q}$ **bulb**. For example $c_{\frac{2}{5}} = \frac{1}{2}e^{\frac{2}{5}2\pi i} - \frac{1}{4}e^{\frac{2}{5}4\pi i}$ is the root point of the $\frac{2}{5}$ bulb.

Note. The derivative of $f_{c_{\frac{2}{5}}} = z^2 + c_{\frac{2}{5}}$ at the fixed point p_- is $e^{\frac{2}{5}2\pi i}$, which suggests that the dynamic of the Julia sets in the $\frac{2}{5}$ bulb may be influenced by the rotation of angle $\frac{2}{5}$. What does this mean? We look at a filled Julia set J_{c_*} corresponding to a point c_* taken from inside the $\frac{2}{5}$ bulb (any Julia set from the bulb can be used, as long as it is inside it). If we follow the orbit of any point from this Julia set it will settle into a rotation cycle with rotation number 2/5. The rotation number means simply that the orbit eventually rotates around the fixed point p_{c_*} at an angle approximately $\frac{2}{5}$.

How to find the external rays to a Julia set from a bulb attached to the main cardioid that meet at the junction point (one of the fixed points) for that set.

The most stable fixed point $p(c_*)$ of the Julia set J_{c_*} is a junction point of its main body and four more similar petals. It is a landing point of five external rays which form a rotation cycle of period $\frac{2}{5}$. To find these external rays we apply the following algorithm.

Divide the interval $(0, 1]$ into two subintervals $I_0 = (0, 3/5]$ and $I_1 = (3/5, 1]$. We will consider the orbit of $\frac{2}{5}$ under rotation angle 2/5. The orbit is, as you already know $\frac{2}{5} \mapsto \frac{4}{5} \mapsto \frac{1}{5} \mapsto \frac{3}{5} \mapsto 1 \mapsto \frac{2}{5} \ldots$. The itinerary of this orbit based on the two intervals I_0 and I_1 defined above is $0100101001\ldots = \overline{01001}$. The binary number determined by the itinerary is $0.\overline{01001} = 9/31$. The number $\frac{9}{31}$ is one of the external rays that land at $p_(c_*)$. All other external rays can be obtained by period doubling and they are $\frac{18}{31}, \frac{5}{31}, \frac{10}{31}, \frac{20}{31}$.

How to find the external rays that land on the $\frac{2}{5}$ bulb of the Mandelbrot set?

We have have exactly two external rays landing at the root of the $\frac{2}{5}$ bulb. One of them is determined by the same algorithm as for the Julia set, that is one of the external rays is the number $\frac{9}{31}$. The other angle can be obtained by switching the last two digits in the main block of the itinerary of $\frac{9}{31}$. Since the binary expansion of $\frac{9}{31}$ is $0.\overline{01001}$ the binary expansion of the other external ray is $0.\overline{01010} = \frac{10}{31}$.

Fact. In general there are exactly two rational rays landing at each $\frac{p}{q}$ bulb. One of these rays, when iterated under period doubling, will give us all of the rays landing at the fixed point of a Julia set from the same bulb.

How to find the external rays that land at the point where the $\frac{p}{q}$ bulb is attached to the main cardioid in the Mandelbrot set.

Here we assume that p and q are mutually prime. The algorithm is similar to the one for the $\frac{2}{5}$ bulb. We follow the $\frac{p}{q}$ rotation cycle of $\frac{p}{q}$ on the unit circle. We construct the itinerary based on the intervals $I_0 = (0, 1 - \frac{p}{q}]$ and $I_1 = (1 - p/q, 1]$ of that period q cycle. The binary expansion that corresponds to this itinerary gives us one of the landing rays. The other landing ray has the same binary expansion, except for the last two digits in the repeating block. If we switch them, that creates a new infinite binary number which is the second landing ray. (Note that now I_0 contains its larger endpoint, not 0 as in the algorithm for working out binary explansions.)

Rotation Cycles

Consider the doubling function on the unit circle. There are exactly six cycles of period five under the doubling function. You should write them down explicitly. Only four of them are rotation cycles. For example

$$\frac{5}{31} \mapsto \frac{10}{31} \mapsto \frac{20}{31} \mapsto \frac{9}{31} \mapsto \frac{18}{31} \mapsto \frac{5}{31}$$

is a rotation cycle and

$$\frac{3}{31} \mapsto \frac{6}{31} \mapsto \frac{12}{31} \mapsto \frac{24}{31} \mapsto \frac{17}{31} \mapsto \frac{3}{31}$$

is not. The first cycle repeats the combinatorial structure of the rotation map on the unit circle of period 2/5. The second cycle does not have any of the combinatorial structures of a rotation on the unit circle of period five.

Definition. Let $\Theta = \{\theta_1, \theta_2 ... \theta_q\}$ be a cycle of period q on the unit circle under the doubling map. Let $0 < p < q$. We say that Θ is a cycle of rotation number $\frac{p}{q}$ if the iterates of θ_1 are ordered on the unit circle like any orbit of the rotation $\frac{p}{q}$ on the unit circle.

Fact All bulbs of period n in the Mandelbrot set have a root point, which is a landing point of exactly two rays of the type $\frac{p_1}{2^n-1}$ and $\frac{p_2}{2^n-1}$. Some bulbs have rotation numbers and some don't. (As we said above, all bulbs attached to the main cardioid do have rotation numbers. You should check this out.) The bulbs that have a rotation number have Julia sets for which the orbits of the points settle into a rotation cycle. This rotation cycle is called the rotation cycle for the corresponding bulb.

Question 5:

(1) Identify all bulbs of period 3 and 4. Determine which bulbs have a rotation number and which don't. Hint: Draw all external rays to the Mandelbrot set of type $\frac{p}{2^3-1}$ and $\frac{p}{2^4-1}$. Look at the combinatorics of the orbits of a Julia set from each of these bulbs.

(2) Make a conjecture about the location of the rotational bulbs with respect to the Mandelbrot set. If needed locate all bulbs of period five. How many are there?

(3.1) Identify a bulb of rotation number $\frac{1}{3}$. Find the external rays that land at the root of the bulb as well as the external rays landing at the fixed point of a Julia set within this bulb.
(3.2) Repeat exercise (3.1) with a bulb of rotation number $\frac{1}{4}$.
(3.3) Find a bulb of rotation number $\frac{1}{2}$. Which is the biggest bulb lying between that bulb and the bulb with rotation number $\frac{1}{3}$? Do you see any relationship between the rotation numbers of the three bulbs? What is the rotation number of the largest bulb lying between the bulb $\frac{1}{2}$ and the bulb $\frac{2}{5}$?