THE HODGE THEORY OF ALGEBRAIC MAPS

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Report on joint work with Luca Migliorini at Bologna on new structures on the cohomology of complex algebraic varieties.

Most results hold for proper maps of algebraic varieties.

For ease of exposition, today I discuss the projective case.

- $f : X \longrightarrow Y$ is a map of projective varieties.
- η ample divisor on X, $\quad L := f^*A$, A ample on Y.
- X smooth of dimension n.
- $H(X) = H(X, \mathbb{Q})$ (if X singular, use $IH(X)$).

SUMMARY

- $f : X \longrightarrow Y$ induces a vector space decomposition

$$\phi \ : \ H^\bullet(X) \ \simeq \ \bigoplus_{i \in \mathbb{Z}} H^\bullet_i(X).$$

- The Hodge Theory of $H^\bullet_\bullet(X)$:
  - (η, L)-Primitive Decomposition:

$$H^\bullet_\bullet(X) = P_{\eta,L} \oplus \langle \mathrm{Im}\ \ \eta, \mathrm{Im}\ \ L \rangle.$$

  - Primitive spaces are polarized via cup product on X.
  - Decomposition by strata $Y_\alpha \subseteq Y$:

$$H^\bullet_\bullet(X) = \bigoplus_\alpha H^\bullet_{\bullet,\alpha}(X).$$

- The Hodge theory of the refined intersection forms on the fibers of $f : X \longrightarrow Y$.
- Nondegeneration of refined intersection forms and the Decomposition Theorem.
- Hodge-theoretic splitting

$$\varphi_\eta \ : \ H^\bullet(X) \ \simeq \ \bigoplus_i H^\bullet_i(X).$$

DELIGNE's THEOREM FOR f SMOOTH

Theorem (Deligne '69) *Let $f : X \to Y$ be smooth.*
There is a non canonical isomorphism

$$Rf_*\mathbb{Q}_X \simeq \bigoplus_i R^i f_*\mathbb{Q}_X[-i].$$

Some consequences:

- The Leray Spectral Sequence of a smooth f degenerates.
- The Theorem of the Fixed Part.
- The Global Invariant Cycle Theorem.

Etc.

f SINGULAR: THE DECOMPOSITION THEOREM

Gelfand and MacPherson conjectured that $Rf_*\mathbb{Q}$ splits into a direct sum of shifted intersection cohomology complexes.

Beilinson, Bernstein, Deligne and Gabber proved this conjecture and introduced perverse sheaves for this purpose.

A perverse sheaf is a special complex of sheaves.

Goresky and MacPherson Intersection Cohomology Complexes are important examples of perverse sheaves.

The key fact is that one splits $Rf_*\mathbb{Q}$ using the new functors:

$$\boxed{{}^pR^i f_* \qquad i\text{-th perverse direct image.}}$$

Decomposition Theorem (BBDG's DT)

$$Rf_*\mathbb{Q}_X[n] \;\simeq\; \bigoplus_i \left({}^pR^i f_*\mathbb{Q}_X[n]\right)[-i],$$

$${}^pR^i f_*\mathbb{Q}_X[n] \quad \textit{is a direct sum of IC's.}$$

The DT is the deepest known result concerning the homology of algebraic varieties.

THE PERVERSE FILTRATION

There are the perverse truncation functors:

$$\ldots \longrightarrow {}^{p}\tau_{\leq i-1} \longrightarrow {}^{p}\tau_{\leq i} \longrightarrow \ldots \longrightarrow Id.$$

When applied to $Rf_*\mathbb{Q}_X[n]$, the truncation functors give rise to the perverse filtration on $H(X)$:

$$\boxed{0 \subseteq \ldots \subseteq H_{\leq i-1}(X) \subseteq H_{\leq i}(X) \subseteq \ldots \subseteq H(X).}$$

The graded pieces

$$\boxed{H^j_i(X) \;:=\; H^j_{\leq i}(X) \,/\, H^j_{\leq i-1}(X)}$$

play an important role.

THE PERVERSE FILTRATION IS HODGE-THEORETIC

Theorem 1 (The perverse filtration is Hodge-theoretic)

$$\boxed{H_{\leq i}(X) \ \subseteq \ H(X) \ \ \textit{Pure Hodge Structure.}}$$

$$\boxed{H^j_i(X) \ \simeq H^j({}^{p}\!R^i f_* \mathbb{Q}_X) \ \ \textit{PHS.}}$$

This seems to be a rather surprising Hodge-theoretic property of the topologically-defined perverse filtration.

It is proved by first proving the following Hard Lefschetz-type result for the action of η and $L = f^*A$ on $H^{\bullet}_{\bullet}(X)$:

$$\boxed{L^k \ : \ H_i^{(n+i)-k}(X) \ \simeq \ H_i^{(n+i)+k}(X),}$$

$$\boxed{\eta^i \ : \ H^j_{-i}(X) \ \simeq \ H^{j+2i}_i(X).}$$

DISPLAY TABLE FOR $H_{\bullet}^{\bullet}$

The Decomposition Theorem gives a non-canonical

$$\boxed{\phi \ : \ H^{\bullet}(X) \ \simeq \ \bigoplus_i H_i^{\bullet}(X)}$$

Example of table: $n = 4$, ${}^{p}\!R^i f_* \mathbb{Q}_X[4] = 0$ for $|i| > 2$.

perversity index	-2	-1	0	$+1$	$+2$
$H^0(X)$			$*$		
$H^1(X)$	$*$	$*$	$*$		
$H^2(X)$	●	$*$	$*$		
$H^3(X)$	$*$	●	$*$	$*$	
$H^4(X)$		$*$	●	$*$	
$H^5(X)$		$*$	$*$	●	$*$
$H^6(X)$			$*$	$*$	●
$H^7(X)$			$*$	$*$	$*$
$H^8(X)$			$*$		

L acts vertically, η acts diagonally.

Each column is like the cohomology of a (non-existing) projective manifold acted upon by an ample line bundle.

Each diagonal slide: similarly, wrt η.

Due to the loss of positivity in pulling-back, the Hard Lefschetz Theorem cannot hold for $L = f^*A$ on $H(X)$:

it is as if the perverse filtration were calibrated with the purpose of restoring Hard Lefschetz for L.

In the following three tables a surface X^2 is mapped onto a surface, a curve and a point, respectively.

As $L = f^*A$ becomes less and less positive, the display becomes more and more slanted.

perversity index	-2	-1	0	$+1$	$+2$
$H^0(X)$			$*$		
$H^1(X)$			$*$		
$H^2(X)$			$\bullet$		
$H^3(X)$			$*$		
$H^4(X)$			$*$		

perversity index	-2	-1	0	$+1$	$+2$
$H^0(X)$		$*$			
$H^1(X)$		$\bullet$	$*$		
$H^2(X)$		$*$	$\bullet$	$*$	
$H^3(X)$			$*$	$\bullet$	
$H^4(X)$				$*$	

perversity index	-2	-1	0	$+1$	$+2$
$H^0(X)$	$\bullet$				
$H^1(X)$		$\bullet$			
$H^2(X)$			$\bullet$		
$H^3(X)$				$\bullet$	
$H^4(X)$					$\bullet$

GENERALIZED HODGE-RIEMANN RELATIONS

– Classical Hard Lefschetz: $\eta^i : H^{n-i}(X) \simeq H^{n+i}(X)$;

– Primitive Decomposition : $H^{n-i}(X) \;=\; P^{-i} \oplus \mathrm{Im}\ \eta$;

– Hodge-Riemann Relations: $\int_X \eta^i \wedge - \wedge -$ polarizes P^{-i}.

There is an analogous picture for η and L acting on $H^{\bullet}_{\bullet}(X)$:

$$H^{n-i-j}_{-i}(X) \;=\; P^{-j}_{-i} \oplus \langle \mathrm{Im}\ \eta, \mathrm{Im}\ L \rangle,$$

where $$P^{-j}_{-i} \;:=\; \mathrm{Ker}\ \eta^{i+1} \cap \mathrm{Ker}\ L^{j+1};$$

the bilinear form

$$\int_X \eta^i \wedge L^j \wedge - \wedge - \qquad \text{on } \; H^{n-i-j}(X)$$

descends to the graded piece $H^{n-i-j}_{-i}(X)$ and

Theorem 2 (Generalized Hodge-Riemann Relations)

The form above polarizes the primitive space P^{-j}_{-i}.

It is proved using a technique we call *approximation of primitive vectors.*

EXAMPLE OF REFINED INTERSECTION FORM

$$\begin{tikzcd}
\mathbb{P}^1 = F_y \arrow[r] \arrow[d] & \widetilde{\mathbb{C}^2} \arrow[d, "f"] \\
y \arrow[r] & \mathbb{C}^2
\end{tikzcd}$$

There is a sequence of maps on $\mathbb{C}^2$:

$$\mathbb{Q} \longrightarrow Rf_*\mathbb{Q} \longrightarrow H^2(F_y)[-2] \xrightarrow{e} \mathbb{Q}[1]$$

and

$$\boxed{Rf_*\mathbb{Q} \;\simeq\; \mathbb{Q} \oplus H^2(F_y)[-2] \qquad \text{IFF} \qquad e = 0.}$$

The key point is that the map e appears in

$$\begin{tikzcd}
\cdots \arrow[r] & H^2(\widetilde{\mathbb{C}^2}, \widetilde{\mathbb{C}^2} \setminus F_y) \arrow[r] \arrow[d, "\simeq"] & H^2(\widetilde{\mathbb{C}^2}) \arrow[r, "e"] \arrow[d, "\simeq"] & H^2(\widetilde{\mathbb{C}^2} \setminus F_y) \arrow[r] & \cdots \\
& H_2(F_y) \arrow[r, "\iota"] & H^2(F_y) & &
\end{tikzcd}$$

The map ι can be re-interpreted as a bilinear form:

$$H_2(F_y) \times H_2(F_y) \xrightarrow{\iota} \mathbb{Q}, \qquad [F] \cdot [F] = -1,$$

the refined intersection form associated with $F_y \subseteq \widetilde{\mathbb{C}^2}$.

Upshot: $\boxed{\iota \text{ nondegenerate} \quad \text{IFF} \quad e = 0 \quad \text{IFF} \quad Rf_*\mathbb{Q} \text{ splits.}}$

REFINED INTERSECTION FORMS

Composing the natural maps below

$$H_{n-j}(F_y) \longrightarrow H_{n-j}(X) \overset{PD}{\simeq} H^{n+j}(X) \longrightarrow H^{n+j}(F_y),$$

one obtains a linear map:

$$\boxed{H_{n-j}(F_y) \xrightarrow{\iota} H^{n+j}(F_y)}$$

or, equivalently, a bilinear map:

$$\boxed{H_{n-j}(F_y) \times H_{n+j}(F_y) \xrightarrow{\iota} \mathbb{Q}.}$$

$$\boxed{\iota \ := \ \text{refined intersection form for } F_y \subseteq X.}$$

N.B. : while in general ι is far from being nondegenerate, its graded counterpart behaves very well.

There are filtrations induced by f on

$$H_\bullet(F_y) \qquad \text{and} \qquad H^\bullet(F_y).$$

The refined intersection form ι is filtered and descends to:

$$\iota \ : \ H_{n-j,i}\,(F_y) \longrightarrow H^{n+j}_i(F_y).$$

Theorem 3 (The Refined Intersection Form Theorem)

$$\iota \ : \ H_{n-j,i}\,(F_y) \times H_{n+j,i}\,(F_y) \longrightarrow \mathbb{Q}$$

is zero if $i \neq j$ and is $\underline{\text{nondegenerate}}$ if $i = j$.

Idea of proof for $H_{n,0}\,(F_y)$

- MHS on $H_n(F_y)$ induces MHS on $H_{n,0}\,(F_y)$.
- $H_{n,0}\,(F_y) \longrightarrow H^n_0(X)$ is mono, hence $H_{n,0}\,(F_y)$ PHS.
- Since L is trivial on F_y :

  $$H_{n,0}\,(F_y) \ \subseteq \ \text{Ker} \ L \ \subseteq \ H^n_0(X)$$

- By the Generalized Hodge-Riemann Relations, the form $\int_X$ polarizes Ker L.

  Its restriction polarizes $H_{n,0}\,(F_y)$ and is nondegenerate.
- This restriction is the refined form ι. □

EXAMPLES OF REFINED INTERSECTION FORMS

Example 1 $X^2 \longrightarrow Y^2$.

$$H_{2,0}(F_y) = H_2(F_y)$$

ι is negative-definite: $\boxed{\text{Grauert's Criterion.}}$

Example 2 $X^2 \longrightarrow Y^1$.

$$H_{2,-1} = \langle [F_y] \rangle, \qquad H_{2,0} = H_2(F_y)/\langle [F_y] \rangle$$

ι is zero on $H_{2,-1}$ and negative-definite on $H_{2,0}$: $\boxed{\text{Zariski Lemma.}}$

Example 3 $X^3 \longrightarrow Y^3$.

$$H_{3,0}(F) = H_3(F)$$

$$\int_X - \wedge - \ : \ H_3(F) \times H_3(F) \longrightarrow \mathbb{Q}$$

new contraction criterion (exemplified here for $n = 3$):

$\boxed{H_3(F) \text{ is a PHS and the self-intersection form polarizes.}}$

THE DECOMPOSITION THEOREM

In the example of $\widetilde{\mathbb{C}^2} \longrightarrow \mathbb{C}^2$:

1) the complex $Rf_*\mathbb{Q}[2] = {}^p\!R^0 f_*\mathbb{Q}[2]$,

2) ι on $H_{2,0}(F_y)$ is nondegenerate and this implies

3) ${}^p\!R^0 f_*\mathbb{Q}[2]$ splits as $IC_{\mathbb{C}^2} \oplus IC_y$.

In general, we show that
the nondegeneration of ι implies that ${}^p\!R^i f_*\mathbb{Q}_X[n]$ splits.

This leads to a geometric proof of the DT based on the nondegeneration of the various forms ι.

Decomposition Theorem

$$Rf_*\mathbb{Q}_X[n] \simeq \bigoplus_i \left({}^p\!R^i f_*\mathbb{Q}_X[n]\right)[-i].$$

$${}^p\!R^i f_*\mathbb{Q}_X[n] = \bigoplus_\alpha IC_{\overline{Y}_\alpha}(\mathcal{L}_{i,\alpha}),$$

where $Y_\alpha \subseteq Y$ is a collection of locally closed smooth subvarieties and the $\mathcal{L}_{i,\alpha}$ are semisimple local systems on Y_α.

The DT was proved by BBDG using positive characteristic.

M. Saito: transcendental proof (of a stronger statement).

dC–Migliorini.

C. Sabbah, T. Mochizuki: split the direct image of semisimple local systems, semisimple perverse sheaves, resp.

<u>SCHEME OF PROOF OF DT</u>

- Induction on

$$r(f) \ := \ \min_{r} \{ \, {}^{p}\!R^{i} f_{*}\mathbb{Q}_{X}[n] = 0, \ \forall |i| > r \ \}.$$

- Use the inductive hypothesis to prove the Relative Hard Lefschetz Theorem:

$$\eta^{i} \ : \ {}^{p}\!R^{-i} f_{*}\mathbb{Q}_{X}[n] \ \simeq \ {}^{p}\!R^{i} f_{*}\mathbb{Q}_{X}[n].$$

- The Deligne-Lefschetz Criterion then implies

$$Rf_{*}\mathbb{Q}_{X}[n] \ \simeq \ \bigoplus_{i} \left({}^{p}\!R^{i} f_{*}\mathbb{Q}_{X}[n]\right)[-i].$$

- Use Relative Lefschetz Theorem on Hyperplane Sections to prove that

$${}^{p}\!R^{i} f_{*}\mathbb{Q}_{X}[n] \quad \text{is semisimple } \forall i \neq 0.$$

- Use the nondegeneration of the refined intersection forms to prove that

$${}^{p}\!R^{0} f_{*}\mathbb{Q}_{X}[n] \quad \text{is semisimple.}$$

HODGE-THEORETIC SPLITTINGS

Theorem 4 *The canonical splitting according to strata* Y_α

$$H_i^\bullet(X) \;=\; \bigoplus_\alpha H_{i,\alpha}^\bullet(X) \qquad PHS.$$

Example $f : X^2 \longrightarrow Y^2$.

$$H^2(X) \;=\; IH^2(Y) \overset{\perp}{\bigoplus} H^2(F_y).$$

$H^2(F_y)$ is a PHS of pure type $(1,1)$.
By orthogonality, $IH^2(Y)$, is PHS.

Let $f : X \longrightarrow Y$ be a resolution.
Then $IH^\bullet(Y)$ is one of the summands of $H_0^\bullet(X)$ and we have

Corollary $IH^\bullet(Y)$ *admits a canonical PHS.*

This is not the same as stating that

$$IH^\bullet(Y) \subseteq H_0^\bullet(X) \longrightarrow H^\bullet(X) \qquad PHS.$$

This can be arranged, again non-canonically, but it requires a separate argument.

The splitting predicted by the DT

$$\phi \;:\; H^\bullet(X) \;\simeq\; \bigoplus_i H_i^\bullet(X)$$

is <u>not</u> necessarily a map of PHS.

Theorem 5 (Hodge-Theoretic splitting of $H(X)$)

The splitting can be realized by a map φ of PHS.

φ depends only on η.

Let $X \longrightarrow Y$ be a resolution of the singularities of Y.

Corollary $\qquad \varphi(IH(Y)) \subseteq H(X) \qquad PHS.$

Every direct summand in the decomposition

$$H_i^\bullet(X) \;=\; \bigoplus_\alpha H_{i,\alpha}^\bullet(X)$$

gives rise to a projector $p = p(\varphi) \in H^{2n}(X \times X)$:

$$p \;:\; H(X) \longrightarrow H_{i,\alpha}^\bullet(X) \longrightarrow H(X).$$

Corollary $\qquad \boxed{p \in H_{\mathbb{Q}}^{n,n}(X).}$

The question of whether or not p is algebraic is difficult: as a special case one gets the Standard Conjectures for the Künneth components.

MOTIVIC SPLITTINGS

Insight coming from the properties of the intersection forms has been used to prove explicit canonical motivic decompositions in some significant cases:

- $S^{[d]} \longrightarrow S^{(d)}$ (Hilbert schemes of d points on a S^2);
- $f : X \to Y$ semismall, i.e. $Rf_*\mathbb{Q}_X[n] = {}^pR^0 f_*\mathbb{Q}_X[n]$;
- $f : X^3 \longrightarrow Y^3$ (resolutions of threefolds).

The point is to show that the projectors p are algebraic.

In his Stony Brook Ph.D dissertation, L. Li is working out a canonical motivic decomposition for Fulton-MacPherson compactifications:

$$X[d] \longrightarrow X^d.$$