

MAT 322 - HW 11

3-35; let the rectangle be  $R = [a_1, b_1] \times \dots \times [a_n, b_n]$ . (It is easy to see that open or closed rectangles ~~make~~ <sup>make</sup> no difference here).

$$\begin{cases} g(e_i) = e_i & i \neq j \\ g(e_j) = \alpha e_j \end{cases} \Rightarrow g(R) = [a_1, b_1] \times \dots \times [\alpha a_j, \alpha b_j] \times \dots \times [a_n, b_n]$$

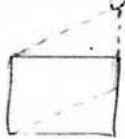
$$\det g = \det \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & \alpha & \\ & & & \ddots \\ 0 & & & & 1 \end{pmatrix} = \alpha$$

Done!

$$\begin{cases} g(e_i) = e_i & i \neq j \\ g(e_j) = e_j + e_k \end{cases}$$

$$\det g = \det \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = 1$$

By the same trick in elementary mathematics, when we proved that the area of parallelogram and rectangle are equal:



you can see that the area of parallelepiped and rectangle are equal.

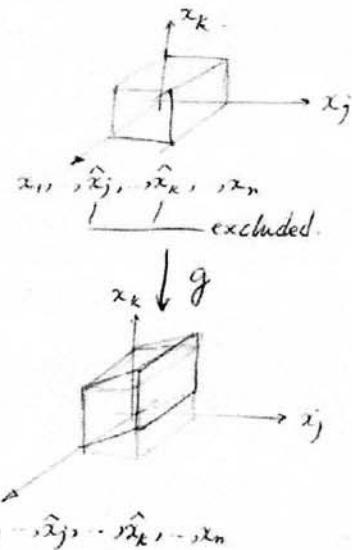
$\triangle$ : You have to be more rigorous than this in the proof!

$$\begin{cases} g(e_k) = e_k & k \neq j \\ g(e_i) = e_j \\ g(e_j) = e_i \end{cases}$$

$$g(R) = [a_1, b_1] \times \dots \times [a_j, b_j] \times \dots \times [a_i, b_i] \times \dots \times [a_n, b_n]$$

$$\det g = \det \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = 1$$

HW 11 (1)



b). We know from linear algebra, that <sup>for</sup> any invertible matrix  $\{g\}$

$$g = g_1 \circ \dots \circ g_m \circ I = g_1 \circ \dots \circ g_m$$

$$\text{for appropriate } g_i\text{'s. } \Rightarrow \det g = \prod_{i=1}^m \det |g_i|$$

If  $g$  is not invertible,  $g(\mathbb{R})$  is a ~~non~~ degenerate rectangle of volume 0.

3-41: a) injectivity is checked easily.

$$\frac{\partial f(r, \theta)}{\partial (r, \theta)} = \begin{bmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{bmatrix} \rightarrow \det Df = r > 0$$

b) You can invert  $Df$ , if you need the inverse in terms of  $(r, \theta)$ , or see, directly, that:

$$\frac{\partial(r, \theta)}{\partial(x, y)} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-1}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix}$$

c) By change of variables, and  $\frac{\partial(x, y)}{\partial(r, \theta)} = r$

$$\int_C h = \int_{r_1}^{r_2} \int_{\theta_1}^{\theta_2} r g \, d\theta dr.$$

Let  $r_1 \rightarrow 0$ ,

$$\int_{B_r} h = \int_0^r \int_0^{2\pi} r g \, d\theta dr. \quad \text{Note that polar coordinates}$$

is not defined at the origin, nevertheless, as  $h$  is integrable on  $B_r$ , the value of one point does not change anything.

$$c) \int_{B_r} e^{-(x^2+y^2)} dx dy = \int_0^{2\pi} \int_0^r r e^{-r^2} dr d\theta = -\pi e^{-r^2} \Big|_0^r = \pi(1 - e^{-r^2})$$

$$d) \int_C e^{-(x^2+y^2)} dx dy = \int_{-r}^r e^{-x^2} \int_{-r}^r e^{-y^2} dy dx = \left( \int_{-r}^r e^{-x^2} dx \right) \left( \int_{-r}^r e^{-y^2} dy \right) \\ = \left( \int_{-r}^r e^{-x^2} dx \right)^2.$$

$$e: 1+d) \int_{C_{r/2}} e^{-(x^2+y^2)} dx dy \leq \int_{B_r} e^{-(x^2+y^2)} dx dy \leq \int_{C_r} e^{-(x^2+y^2)} dx dy.$$

not?

$$\text{So } \lim_{r \rightarrow \infty} \int_{C_r} \dots = \lim_{r \rightarrow \infty} \int_{B_r} \dots \Rightarrow \pi(1 - e^{-r^2}) \xrightarrow{r \rightarrow \infty} \pi \\ \Rightarrow \left( \int_{-r}^r e^{-x^2} dx \right)^2 = \pi$$

Splendid! Isn't it?

3-40: Revisited:

I shall give two solutions. The first one is the very proof given in the previous post, clarified by a change of notation and some more details.

1) Assume  $g(0) = 0$ . We proceed inductively. Assume we've got a function  $I_j(x^1, \dots, x^n) = (x^1, \dots, x^{j-1}, p^j, \dots, p^n)$ .

Note that  $g(\cdot)$  is of the form  $I_0 = (g^1, \dots, g^n)$ .

$$\rightarrow D I_j = \left( \begin{array}{c|c} I_{(j-1) \times (j-1)} & 0 \\ \hline D_j p^j & \\ \vdots & \\ D_j p^n & \end{array} \right)$$

Moreover, assume  $I_j(0) = 0$ , and  $D I_j(0)$  is non-singular. Then,

$$(D I_j(0))(\hat{e}_j) = j\text{-th column of } D I_j(0) \neq 0$$

by  $D I_j$  being non-singular.

Note that  $j$ -th column of  $D I_j(0) = \left( \begin{array}{c} 0 \\ \vdots \\ 0 \\ D_j p^j \\ \vdots \\ D_j p^n \end{array} \right) \neq 0$ .

$\Rightarrow$  for some  $k: j \leq k \leq n: D_j p^k \neq 0$ .

Define  $g_j(x^1, \dots, x^n) = (x^1, \dots, x^j, p^k, x^{j+1}, \dots, x^n)$

$$\rightarrow D g_j(0) = \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & D_j p^k & & \\ 0 & & \vdots & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} \rightarrow \text{By } D_j p^k \neq 0 \rightarrow$$

$\det(D g_j(0)) \neq 0 \Rightarrow g_j(\cdot)$  is, by Inverse function thm, invertible (locally).

3-40 - Con'd:

We constructed  $g_j$  by  $\Gamma_j$ . Let  $T_j$  be a linear transformation which exchanges 'k' and 'j'.

And  $(g_j)^{-1} = (x^1, \dots, x^j, q^k, x^{j+1}, \dots, x^n)$ .

[It is easy to verify that  $(g_j)^{-1}$  has to be of this form].

$\rightarrow \Gamma_j \circ g_j^{-1}(x^1, \dots, x^n) = (x^1, \dots, x^{j-1}, p^j, \dots, \overset{\substack{\text{k-th place} \\ \downarrow}}{x^k}, \dots, p^n)$

$\Rightarrow T_j \circ \Gamma_j \circ g_j^{-1}(x^1, \dots, x^n) = (x^1, \dots, x^{j-1}, x^j, p^{j+1}, \dots, p^j, \dots, p^n)$

Now,  $T_j \circ \Gamma_j \circ g_j^{-1}$  is of the form  $\Gamma_{j+1}$   $\Rightarrow$

Note that  $g \neq \Gamma_0$   $\hat{=}$  admissible for

Define  $\Gamma_{j+1} := T_j \circ \Gamma_j \circ g_j^{-1} \Rightarrow T_j^{-1} \circ \Gamma_{j+1} \circ g_j = \Gamma_j$

$\rightarrow$  Note that  $g = \Gamma_0 \Rightarrow g = \Gamma_0 = T_0^{-1} \circ \Gamma_1 \circ g_0 =$

$T_0^{-1} \circ (T_1^{-1} \circ \Gamma_2 \circ g_1) \circ g_0 = T_0^{-1} \circ T_1^{-1} \circ T_2^{-1} \circ \Gamma_3 \circ g_2 \circ g_1 \circ g_0 = \dots$

$\dots = \underbrace{T_0^{-1} \circ \dots \circ T_{n-1}^{-1}}_T \circ \underbrace{\Gamma_{nm}}_{\tilde{g}_n} \circ g_{n-1} \circ \dots \circ g_1 \circ g_0$

which is the desired form.

Method II: This is gotten from ~~your~~ <sup>some of</sup> students' homework.

Assume  $Dg(a) = I$ , if not, replace  $g(\cdot)$  by  $(Dg(a))^{-1} \circ g$

If  $g = (h^1, \dots, h^n)$ , define:

$$g_1(x^1, \dots, x^n) = (h^1, x^2, \dots, x^n)$$

$$Dg_1(a) = \begin{pmatrix} 1 & & 0 \\ 0 & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix} Dh^1 = (1, 0, \dots, 0) \text{ by assumption}$$

$\Rightarrow g_1$  is locally invertible.  $\Rightarrow$

$$(g_1)^{-1} = (\phi^1, x^2, \dots, x^n).$$

Define  $\Gamma_1 = (g \circ (g_1)^{-1})^{-1} \Rightarrow$

$$\Gamma_1(x^1, \dots, x^n) = (x^1, h^2, \dots, h^n).$$

~~neglecting the first~~ Apply the same to  $\Gamma_1$  to

get  $\Gamma_2 = (x^1, x^2, \tilde{h}^3, \dots, \tilde{h}^n)$  and so on.   
 such that  $\Gamma_2 = \Gamma_1 \circ (g_2)^{-1}$

$$g = \Gamma_1 \circ g_1 = \Gamma_2 \circ g_2 \circ g_1 = \dots = g_n \circ \dots \circ g_1$$