1) Cauchy Convergence Criterion: A sequence (x_n) is Cauchy if and only if it is convergent.

Proof. Suppose (x_n) is a convergent sequence, and $\lim(x_n) = x$. Let $\epsilon > 0$. We can find $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - x| < \epsilon/2$. Therefore, by the triangle inequality, for all $m, n \geq \mathbb{N}$, $|x_m - x_n| \leq |x_m - x| + |x - x_n| < \epsilon/2 + \epsilon/2 = \epsilon$. So (x_n) is Cauchy.

Conversely, suppose (x_n) is Cauchy. Let $\epsilon > 0$. By a result proved in class, (x_n) is bounded. By Bolzano-Weierstrass, it has a convergent subsequence (x_{n_k}) with $\lim(x_{n_k}) = x$ for some x. We can find $K \in \mathbb{N}$ such that for all $k \geq K$, $|x_{n_k} - x| < \epsilon/2$. We can also find M such that for all $m, n \geq M$, $|x_m - x_n| < \epsilon/2$. Let $N = \sup\{K, M\}$. Then since $n_k \geq k$ for all k, if $k \geq N$, we have that $k, n_k \geq M$ and $n_k \geq K$. Therefore, for all $k \geq N$, $|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \epsilon/2 + \epsilon/2 = \epsilon$ by the Triangle Inequality. Therefore, (x_n) is Cauchy. □

2) (a) There is no example. Suppose (x_n) is monotone and bounded. Then by the Monotone Convergence Theorem, it converges. Therefore, (x_n) is not divergent. Therefore, (x_n) is not properly divergent.

(b) Let (x_n) be the sequence given by $x_n = \begin{cases} 1 + 1/n & \text{if n is even} \\ 1/n & \text{if n is odd} \end{cases}$

The even-indexed subsequence converges to 1 and the odd subsequence converges to 0.

(c) $((-1)^n)$ is a bounded sequence. It is a subsequence of itself, and it diverges.

(d) Let (y_n) be the sequence given by $y_n = \begin{cases} n & \text{if n is even} \\ 0 & \text{if n is odd} \end{cases}$

It is unbounded because the even-indexed subsequence is properly divergent. The odd-indexed sequence is constant, and so bounded.

3) $Let \epsilon_0 = 1$, and let $H \in \mathbb{N}$ be arbitrary. Let $n = H$ and $m = H + 1$ so that $n, m \geq H$. We will show that $|x_m - x_n| > \epsilon_0$ (where $x_n = \pi^n$) so that the sequence is not Cauchy. Indeed, since $\pi > 3$, we see that

$$|x_m - x_n| = |\pi^{n+1} - \pi^n| = \pi^n|\pi - 1| > 3^n \cdot 2 > 1.$$

4) (a) divergent

(b) convergent

(c) convergent

(d) convergent

5) (a)

$$\lim \frac{(2 + \frac{1}{n})^n}{2^{n+2}} = \lim \frac{2^n(1 + \frac{1}{2n})^n}{2^{n+2}} = \lim \frac{(1 + \frac{1}{2n})^n}{4} = \frac{1}{4} \lim(1 + \frac{1}{2n})^n$$

Note that $(1+\frac{1}{2n})^{2n}$ is a subsequence of $(1+\frac{1}{n})^n$ (take $n_k = 2k$). Since the latter converges to e, so does the former. Therefore, $\lim(1+\frac{1}{2n})^n = \lim\sqrt{(1+\frac{1}{2n})^{2n}} = \sqrt{\lim(1+\frac{1}{2n})^{2n}} = \sqrt{e}$. Therfore, the desired limit is

$$\frac{\sqrt{e}}{4}$$

(b) We first prove the following: if $A \subseteq \mathbb{R}$, $c \in \mathbb{R}$ is a cluster point of A, $f : A \to \mathbb{R}$ and $f(x) \geq 0$ for all $x \in A$. Furthermore, we assume that $\lim_{x\to c} f$ exists. Then $\lim_{x\to c}\sqrt{f}$ exists, and is equal to $\sqrt{\lim_{x\to c} f}$.

Proof. $\lim_{x\to c} f \geq 0$ by a result in class, so $\sqrt{\lim_{x\to 0} f}$ exists. Let (x_n) be an arbitrary sequence in A that converges to c such that $c \notin \{x_n | n \in \mathbb{N}\}$. Then $(f(x_n))$ is a nonnegative sequence that converges to $\lim_{x\to c} f$. Therefore, by a result in class about sequences, $(\sqrt{f(x_n)})$ that converges to $\sqrt{\lim_{x\to c} f}$. Thus, by the sequential criterion for limits, $\lim_{x\to c}\sqrt{f}$ exists, and is equal to $\sqrt{\lim_{x\to c} f}$ as claimed. □

For the problem, Since we are interested in x near 0, we may restrict our attention to the interval $A := (-1/2, 1/2)$. Thus, $1+x$ and $1-x$ are positive on this interval, and so we calculate (using the above claim in the last step):

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x-x^2} = \lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x-x^2}\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$$

$$= \lim_{x\to 0}\frac{2x}{(x-x^2)(\sqrt{1+x}+\sqrt{1-x})} = \lim_{x\to 0}\frac{2}{(1-x)(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \frac{2}{\lim_{x\to 0}(1-x)(\lim_{x\to 0}\sqrt{1+x}+\lim_{x\to 0}\sqrt{1-x})} = \frac{2}{1(1+1)} = 1$$

6) Let $\epsilon > 0$. Set $M = |c|+1 > 0$. Set $\delta = \inf\{1, \frac{\epsilon}{M^2+M|c|+|c|^2}\} > 0$. We see that $|x^3-c^3| = |x-c||x^2+xc+c^2| \leq |x-c|(|x|^2+|x||c|+|c|^2)$. If $|x-c| < \delta$, then $|x-c| < 1$, and so $||x|-|c|| \leq |x-c| < 1$; implying that $|x| < |c|+1 = M$. Therefore, $|x^3-c^3| < |x-c|(M^2+M|c|+|c|^2) < \frac{\epsilon}{M^2+M|c|+|c|^2}(M^2+M|c|+|c|^2) = \epsilon$. Therefore, $\lim_{x\to c} x^3 = c^3$.