MAT 319 SPRING 2008
MIDTERM I

!!! WRITE YOUR NAME, STUDENT ID BELOW !!!

NAME : ID :

THERE ARE SIX (6) PROBLEMS. THEY HAVE THE INDICATED VALUE.

SHOW YOUR WORK!!!

DO NOT TEAR-OFF ANY PAGE

NO CALCULATORS NO PHONES

ON YOUR DESK: ONLY test, pen, pencil eraser.

1		40pts
2		40pts
3		40pts
4		50pts
5		40pts
6		40pts
Total		250pts

!!! WRITE YOUR NAME, STUDENT ID AND LECTURE N. BELOW !!!

NAME : ID :

LECTURE N.

1. Prove that the number $\sqrt[3]{12}$ is irrational.

Solution. By contradiction, assume $\sqrt[3]{12} = r/s$, where r, s are coprime natural numbers. We get $12s^3 = r^3$. Then 2 divides r, so that $r = 2r_1$. We get $2^2 \cdot 3s^3 = 2^3 r_1^3$. We get $3s^3 = 2r_1^3$, so that 2 divides s. This contradicts r, s coprime.

2. In each case provide an example or state that it is not possible to do so and explain why.

2a Give one example of a countable set that is not finite.
2b Give one example of a countable set that is not denumerable.
2c Give one example of an uncountable set.
2d Give one example of an uncountable set that is not infinite.

Solution.
2a: $\mathbb{N}$.
2b: the empty set.
2c: the set of subsets of $\mathbb{N}$, or the set $\mathbb{R}$.
2d: it is not possible: uncountable means not countable, countable means finite or denumerable; it follows that uncountable means infinite and not denumerable.

3. Assume the triangle inequality

$$|a+b| \leq |a| + |b|, \quad \forall\, a, b \in \mathbb{R}$$

and prove that

$$||a| - |b|| \leq |a-b| \quad \forall\, a, b \in \mathbb{R}.$$

Solution.

$a = (a{-}b){+}b \Longrightarrow |a| = |(a{-}b){+}b| \leq |a{-}b|{+}|b|, \qquad \textit{or} \quad |a| \leq |a{-}b|{+}|b|.$

Subtract $|b|$:

$$|a| - |b| \leq |a-b| \qquad (*).$$

Repeat for b :

$b = (b{-}a){+}a \Longrightarrow |b| = |(b{-}a){+}a| \leq |b{-}a|{+}|a|, \qquad \textit{or} \quad |b| \leq |b{-}a|{+}|a|,$

which re-arranged gives

$$-|a-b| \leq |a| - |b| \qquad (**)$$

By combining $(*)$ with $(**)$, we get the result.

4. For this number 4, just provide the examples (no justification is needed).

4a. Give one example of a subset $A \subseteq \mathbb{R}$ which has a supremum but not an infimum.

4b. Give one example of a subset $B \subseteq \mathbb{R}$ which has an infimum, but not a minimum.

4c. Give an example of a set $C \subseteq \mathbb{R}$ which is denumerable, not bounded below with a supremum that is not a maximum.

4d. Give an example of a set $D \subseteq \mathbb{R} \setminus \mathbb{Q}$ (the irrational numbers) for which $\sup D = 0$ is not a maximum.

Solution.
4a: $A = (-\infty, 0]$.
4b: $B = (0, 1]$.
4c: $C =$ the strictly negative rational numbers.
4d: $D =$ the negative irrational numbers.

5. Prove that a subset $S \subset \mathbb{R}$ is unbounded if and only if it is not contained in any closed bounded interval.

Solution.

We need to prove both directions.

Assume that S is unbounded, but contained in some closed and bounded interval: $S \subseteq [a, b]$, for some $a < b \in \mathbb{R}$. Then $\forall s \in S$, $s \leq b$. This means that the set S is bounded above. Similarly, it is bounded below. It follows that it is bounded. Contradiction.

Assume that S is not contained in any closed and bounded interval, but that it is bounded. By definition of bounded, there are $\alpha < \beta \in \mathbb{R}$ such that $\forall s \in S$, $\alpha \leq s \leq \beta$. This means that $S \subseteq [\alpha, \beta]$, a contradiction.

6. Determine:

$$\lim_{n\to\infty} \left((n!)^{\frac{1}{n^3}} \right).$$

(You may use that $\lim (n^{1/n}) = 1$ and the usual inequalities concerning radicals.)

Solution.

We have $1 \leq n!$ so that, taking roots

$$1 \leq (n!)^{\frac{1}{n^3}} \qquad (*).$$

We have $n! \leq n^n \leq n^{n^2}$, so that, by taking roots we obtain

$$(n!)^{\frac{1}{n^3}} \leq (n^{n^2})^{\frac{1}{n^3}} = n^{\frac{1}{n}}, \ or \qquad (n!)^{\frac{1}{n^3}} \leq n^{\frac{1}{n}} \qquad (**).$$

By combining $(*)$ and $(**)$, we get

$$1 \leq (n!)^{\frac{1}{n^3}} \leq n^{\frac{1}{n}}$$

and the Squeeze Theorem implies that the wanted limit is 1.

7.

8.

9.

10.