$(A^t)_{ij} := A_{ji}$; $A=\begin{pmatrix}1&2\\3&4\\5&6\end{pmatrix}$ $A^t = \begin{pmatrix}1&3&5\\2&4&6\end{pmatrix}$; $A=\begin{pmatrix}1&2\\3&4\end{pmatrix}$ $A^t=\begin{pmatrix}1&3\\2&4\end{pmatrix}$.

FACT: $(A+B)^t = A^t + B^t$.

Proof: $((A+B)^t)_{ij} = (A+B)_{ji} = A_{ji}+B_{ji} = A^t + B^t$. □

DEF: A $n\times n$ is symmetric if $A=A^t$ ($A_{ij}=A_{ji}, \forall i,j$).

FACT $W := \{A\in M_{n\times n} \mid A=A^t\} \subseteq M_{n\times n}$ is a subspace.

Proof: a $O\in M_{n\times n}$ is symmetric $0=O_{ij}=O_{ji}=0$.

b A,B symmetric implies $A+B$ symmetric, in fact we have $(A+B)^t = A^t+B^t = A+B$.

c A symmetric and $c\in F$, then cA is symmetric, in fact $(cA)_{ij} = cA_{ij} = cA_{ji} = (cA)^t$. □

DEF A $n\times n$ is anti-symmetric if $A^t=-A$ ($A_{ij}=-A_{ji}, \forall i,j$).

EXERCISE Anti-symmetric matrices form a subspace of $M_{n\times n}$.

EXERCISE Every $B\in M_{n\times n}(\mathbb{Q})$ can be written in a unique way as $B = S + A$ with $S=S^t$, $A=-A^t$.

(Hint $B+B^t$, $B-B^t$).

$A\in M_{n\times n}$. $tr(A) := \sum_{i=1}^{n} A_{ii}$.

EXERCISE
$$\begin{cases} tr(A+B) = tr(A) + tr(B) \\ tr(cA) = c\, tr\, A \\ tr(AB) = tr(BA) \end{cases}$$

FACT $W := \{A \mid tr(A)=0\} \subset M_{n\times n}(F)$ is a subspace.

Proof a $tr(O) = \sum_{i=1}^{n} 0 = 0$

If $tr A = tr B = 0$, then $tr(A+B) = tr(A)+tr(B) = 0+0=0$

c If $tr(A)=0$, then $tr(cA) = c\, tr(A) = c\cdot 0=0$. □