$\boxed{2.4}$ **#17.**

(a) $0_V \in V_0$ (since V_0 subspace) $\Longrightarrow$ $0_W = T(0_V) \in T(V_0)$.
Let $w_1, w_2 \in T(V_0)$, and $c \in F$. Then
$\exists!\, v_1, v_2 \in V_0$ s.t. $T(v_1) = w_1$, $T(v_2) = w_2$, and:
$w_1 + cw_2 = T(v_1) + cT(v_2) = T(v_1 + cv_2) \in T(V_0)$
since $v_1 + cv_2 \in V_0$ because V_0 is subspace.

(b) Let $\beta = \{v_1, \ldots, v_r\}$ be a basis for V_0.
Then $\{T(v_1), \ldots, T(v_r)\}$ generates $R(T) = T(V_0)$.
Since T is 1-1, $\{T(v_1), \ldots, T(v_r)\}$ is also a L.I. set:
If $a_1 T(v_1) + \ldots + a_r T(v_r) = 0$ $\exists\, a_1, \ldots, a_r \in F$,
$T(a_1 v_1 + \ldots + a_r v_r) = \sum_{i=1}^{r} a_i T(v_i) = 0 \Longrightarrow \left(\frac{\text{since}}{T\ 1\text{-}1}\right)$
$\Longrightarrow \sum_{i=1}^{r} a_i v_i = 0 \underset{\substack{(\text{since} \\ \beta \text{ basis})}}{\Longrightarrow} a_i = 0 \ \forall i = 1, \ldots, r.$

Therefore $T(\beta)$ is a basis for $T(V_0)$, and
thus $\dim T(V_0) = \# T(\beta) = r = \# \beta = \dim V_0$.

<u>Remark</u>: Can you relax the hypothesis on (a) and or (b) for them to still be true?

$\boxed{2.5}$ **#2.**

(b) $Q = [I_{\mathbb{R}^2}]_{\beta'}^{\beta}$

$\begin{pmatrix} 0 \\ 10 \end{pmatrix} = a \begin{pmatrix} -1 \\ 3 \end{pmatrix} + b \begin{pmatrix} 2 \\ -1 \end{pmatrix} \Longrightarrow \begin{cases} -a + 2b = 0 \longrightarrow a = 2b \\ 3a - b = 10 \longrightarrow 6b - b = 10 \end{cases}$
$b = 2$
$a = 4$

$\begin{pmatrix} 5 \\ 0 \end{pmatrix} = a \begin{pmatrix} -1 \\ 3 \end{pmatrix} + b \begin{pmatrix} 2 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} -a + 2b = 5 \longrightarrow -a + 6a = 5 \\ 3a - b = 0 \longrightarrow b = 3a \end{cases}$
$a = 1,\ b = 3$

So $Q = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.

(c) $\begin{pmatrix}1\\0\end{pmatrix} = a\begin{pmatrix}2\\5\end{pmatrix} + b\begin{pmatrix}-1\\-3\end{pmatrix} \Rightarrow \begin{cases} 2a-b=1 \to b = 2a-1 \\ 5a-3b=0 \to \end{cases}$

$\to 5a-6a+3=0 \to a=3$

$b=5$

$\begin{pmatrix}0\\1\end{pmatrix} = a\begin{pmatrix}2\\5\end{pmatrix} + b\begin{pmatrix}-1\\-3\end{pmatrix} \Rightarrow$

$\Rightarrow \begin{cases} 2a-b=0 \to b=2a \\ 5a-3b=1 \to 5a-6a=1 \to a=-1, \ b=-2 \end{cases}$

So $Q = \begin{pmatrix} 3 & -1 \\ 5 & -2 \end{pmatrix}$.

$\boxed{2.6}$ **#14.** We need to show that $\{f_{k+1}, \ldots, f_n\}$ basis for W°.

$W^\circ = \{f \in V^* \mid f(w) = 0 \ \forall w \in W\}$.

First we need to check that $\{f_{k+1}, \ldots, f_n\} \subset W^\circ$:

If $w \in W$, then $w = \sum_{i=1}^{k} a_i x_i$ $\exists a_1, \ldots, a_k \in F$

(since $\{x_1, \ldots, x_k\}$ is a basis for W);

Thus, for $j = k+1, \ldots, n$ fixed,

$f_j(w) = f_j\left(\sum_{i=1}^{k} a_i x_i\right) = \sum_{i=1}^{k} a_i f_j(x_i) = 0$

since $f_j(x_i) = 0 \ \forall j \neq i$.

Since this works $\forall w \in W$, we have $f_j \in W^\circ$ by definition of W°, $\forall j = k+1, \ldots, n$.

Now we can prove $\{f_{k+1}, \ldots, f_n\}$ is a basis for W°.

$\{f_{k+1}, \ldots, f_n\} \subset \beta^*$ and β^* is L.I. set; therefore $\{f_{k+1}, \ldots, f_n\}$ is L.I. too.

To show it generates W°, consider any $f \in W^\circ$:

Since $f \in V^*$ and β^* basis, we can write
$f = \sum_{i=1}^{n} a_i f_i \quad \exists\, a_1, \dots, a_n \in F$.
Since $\{x_1, \dots, x_k\} \subset W$, and $f \in W^\circ$, we have
$f(x_j) = 0 \quad \forall j = 1, \dots, k$.
But $f(x_j) = \sum_{i=1}^{n} a_i f_i(x_j) = \sum_{i=1}^{n} a_i \delta_{ij} = a_j$.
Thus $a_j = 0 \quad \forall j = 1, \dots, k$, i.e.
$f = a_{k+1} f_{k+1} + \dots + a_n f_n$.
So $\dim W^\circ = n-k$, and $\dim W + \dim W^\circ =$
$= k + (n-k) = n = \dim V$.

$\boxed{3.2}$ <u>#6d</u>: Let β, γ be standard bases for $\mathbb{R}^3$ and $P_2(\mathbb{R})$. Then:

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

T invertible $\iff [T]_\beta^\gamma$ invertible $\iff$
$\iff [T]_\beta^\gamma$ has rank 3.
Reducing <u>by rows and columns:</u>

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \longrightarrow$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix} \text{ has } rk = 3.$$

So T invertible. To find its inverse:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & & \\ 1 & -1 & 1 & & 1 & \\ 1 & 0 & 0 & & & 1 \end{array}\right) \xrightarrow[\text{by rows}]{\text{reducing}} \left(\begin{array}{ccc|ccc} 1 & & & 0 & 0 & 1 \\ & 1 & & 1/2 & -1/2 & 0 \\ & & 1 & 1/2 & 1/2 & -1 \end{array}\right)$$

Therefore $T^{-1}: P_2(\mathbb{R}) \longrightarrow \mathbb{R}^3$

$$T^{-1}(b_1 + b_2 x + b_3 x^2) = \left(b_3, \tfrac{1}{2}b_1 - \tfrac{1}{2}b_2, \tfrac{1}{2}b_1 + \tfrac{1}{2}b_2 - b_3\right).$$

$\boxed{3.3}$ <u>#7d</u>: $Ax = b$ has a solution $\iff rk(A) = rk(A|b)$.

$$[A|b] = \left[\begin{array}{cccc|c} 1 & 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 & 1 \\ 4 & 1 & 8 & -1 & 0 \end{array}\right] \longrightarrow$$

(reducing <u>by rows</u>)

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & & & & 25/6 \\ & 1 & & & 4/3 \\ & & 1 & & -5/2 \\ & & & 1 & -2 \end{array}\right].$$

Thus $rk(A) = 4 \geq rk(A|b) \geq rk(A) \Rightarrow$

$\Rightarrow rk(A) = rk(A|b)$ so the system has solution.

$\boxed{3.4}$ <u>#9</u>: With respect to standard coordinates, we obtain the set $\left\{\begin{pmatrix}0\\-1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\2\\3\end{pmatrix}, \begin{pmatrix}2\\1\\1\\9\end{pmatrix}, \begin{pmatrix}1\\-2\\-2\\4\end{pmatrix}, \begin{pmatrix}-1\\2\\2\\-1\end{pmatrix}\right\}$

in $\mathbb{R}^4$.

Consider $A = \begin{pmatrix} 0 & 1 & 2 & 1 & -1 \\ -1 & 2 & 1 & -2 & 2 \\ -1 & 2 & 1 & -2 & 2 \\ 1 & 3 & 9 & 4 & -1 \end{pmatrix}$.

The reduced row echelon form will tell us exactly which columns are L.I., and therefore a basis (since we know that the given sets generates).

$$A \xrightarrow[\text{rref}]{} \begin{bmatrix} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus $\left\{\begin{pmatrix}0 & -1\\-1 & 1\end{pmatrix}, \begin{pmatrix}1 & 2\\2 & 3\end{pmatrix}, \begin{pmatrix}1 & -2\\-2 & 4\end{pmatrix}\right\}$ is a basis for W.