$\boxed{1.4}$ **#9**: $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(F)$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$\boxed{1.5}$ **#19**: Consider a linear combination:

$c_1 A_1^t + c_2 A_2^t + \dots + c_k A_k^t = O$. Then:

$$O = \sum_{i=1}^{k} c_i A_i^t = \left(\sum_{i=1}^{k} c_i A_i\right)^t \Rightarrow \sum_{i=1}^{k} c_i A_i = O \Rightarrow$$

$\Rightarrow c_i = 0 \ \forall i = 1, \dots, k.$

since $\{A_1, \dots, A_k\}$ is linearly indep.

$\boxed{1.6}$ **#7**: $u_1 \neq 0 \Rightarrow \{u_1\}$ is linearly independent.

$$\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = c\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} 2c = 1 \longrightarrow c = \frac{1}{2} \\ -3c = 4 \\ c = -2 \end{cases} \quad \text{no solution}$$

Thus $\{u_1, u_2\}$ is linearly independent.

$u_3 = -4u_1$.

$$\begin{pmatrix} 1 \\ 37 \\ -17 \end{pmatrix} = a\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + b\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} \Rightarrow \begin{cases} 2a + b = 1 \\ -3a + 4b = 37 \\ a - 2b = -17 \end{cases} \longrightarrow$$

$$\longrightarrow \begin{cases} b = 1 - 2a \longrightarrow b = 7 \\ -3a + 4 - 8a = 37 \longrightarrow -11a = 33 \longrightarrow a = -3 \\ a - 2 + 4a = -17 \longrightarrow 5a = -15 \longrightarrow a = -3 \end{cases}$$

So $u_4 = -3u_1 + 7u_2$. $\{u_1, \dots, u_5\}$ spans $\mathbb{R}^3$, and since $\dim \mathbb{R}^3 = 3$, $u_3, u_4 \in \operatorname{Span}\{u_1, u_2\}$, we can conclude that $u_5 \notin \operatorname{span}\{u_1, u_2\}$ and $\{u_1, u_2, u_5\}$ is a basis.

$\boxed{2.1}$ **#4**: $N(T) = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \in M_{2\times 3}(F) \,\middle|\, \begin{matrix} 2a_{11} - a_{12} = 0 \\ a_{13} + 2a_{12} = 0 \end{matrix} \right\}$

$$= \left\{ A \in M_{2\times 3}(F) \,\middle|\, \begin{matrix} a_{12} = 2a_{11} \\ a_{13} = -4a_{11} \end{matrix} \right\} =$$

$$= \left\{ \begin{pmatrix} a_{11} & 2a_{11} & -4a_{11} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \,\middle|\, a_{11}, a_{21}, a_{22}, a_{23} \in F \right\}.$$

A basis for $N(T)$ is:

$$\left\{\begin{pmatrix}1 & 2 & -4\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\right\}.$$

Thus $\text{nullity}(T) = 4$. By rank/nullity thm.,
$\text{rank}(T) = \dim M_{2\times 3}(F) - \text{nullity}(T) = 6 - 4 = 2$.

$R(T) = \text{Span}\{T(E^{11}), T(E^{12}), T(E^{13}), T(E^{21}), T(E^{22}), T(E^{23})\}$.

$T(E^{11}) = \begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}$, $T(E^{12}) = \begin{pmatrix}-1 & 2\\ 0 & 0\end{pmatrix}$ are linearly independent. Thus (since $\text{rank}(T) = 2$):

$\left\{\begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}-1 & 2\\ 0 & 0\end{pmatrix}\right\}$ is a basis for $R(T)$.

T is 1-1 $\Longleftrightarrow N(T) = \{0\}$. Since $\text{nullity}(T) = 4$, T is <u>not</u> 1-1.

T is onto $\Longleftrightarrow R(T) = M_{2\times 2}(F)$. Since $\text{rank}(T) = 2 < 4 = \dim M_{2\times 2}(F)$, T is not onto
$(R(T) \subsetneq M_{2\times 2}(F))$.

$\boxed{2.2}$ <u>#4</u>: $T\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = 1$, $T\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = 1 + x^2$

$T\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = 0$, $T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = 2x$.

$$\left[T\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right]_\gamma = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \left[T\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right]_\gamma = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$\left[T\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right]_\gamma = \begin{pmatrix}0\\0\\0\end{pmatrix} \qquad \left[T\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right]_\gamma = \begin{pmatrix}0\\2\\0\end{pmatrix}$$

Thus $[T]_\beta^\gamma = \begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 0 & 0 & 2\\ 0 & 1 & 0 & 0\end{bmatrix}$.

$\boxed{2.3}$ #9: Let $T : F^2 \longrightarrow F^2$ be defined by:

$T(1,0) = (0,0)$ i.e. $T(a,b) = (0,b)$.

$T(0,1) = (0,1)$

Let $U : F^2 \longrightarrow F^2$ be defined by:

$U(1,0) = (0,1)$ i.e. $U(a,b) = (0,a)$.

$U(0,1) = (0,0)$

Then $(UT)(a,b) = U(T(a,b)) = U(0,b) = (0,0)$

and $(TU)(a,b) = T(U(a,b)) = T(0,a) = (0,a)$

$\forall (a,b) \in F^2$.

Let $\beta = \{(1,0),(0,1)\}$ standard basis of F^2.

Then if $B := [T]_\beta = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and

$A := [U]_\beta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, we have:

$BA = [T]_\beta \cdot [U]_\beta = [T \circ U]_\beta \neq 0$ since $T \circ U \neq T_0$

$AB = [U]_\beta \cdot [T]_\beta = [U \circ T]_\beta = [T_0]_\beta = 0$.