$\boxed{4.1}$

(5) Say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$.

$\det(B) = cb - ad = -(ad - bc) = -\det(A)$.

(6) $A = \begin{bmatrix} a & a \\ b & b \end{bmatrix}$. $\det(A) = ab - ba = 0$.

(10)(a) $AC = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix} =$

$$= \begin{bmatrix} A_{11}A_{22} - A_{12}A_{21} & -A_{11}A_{12} + A_{11}A_{12} \\ A_{21}A_{22} - A_{21}A_{22} & -A_{12}A_{21} + A_{11}A_{22} \end{bmatrix} =$$

$$= \begin{bmatrix} \det(A) & 0 \\ 0 & \det(A) \end{bmatrix} = \det(A) \cdot I$$

Similarly for CA.

(b)

$\det(C) = A_{22}A_{11} - (-A_{21}) \cdot (-A_{12}) = A_{11}A_{22} - A_{12}A_{21} =$
$= \det(A)$

(c) $A^t = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} \Rightarrow$ its adjoint is: $\begin{bmatrix} A_{22} & -A_{21} \\ -A_{12} & A_{11} \end{bmatrix} =$

$$= \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}^t = C^t$$

(d) Clear by thm. 4.2.

(11) By (i), $\delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \delta\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} + \delta\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} =$

$$= \delta\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} + \delta\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + \delta\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} + \delta\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} =$$

$$= ac\,\delta\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + ad\,\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + bd\,\delta\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + bc\,\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

By (ii), $\delta\begin{pmatrix}1&0\\1&0\end{pmatrix}=0$, $\delta\begin{pmatrix}0&1\\0&1\end{pmatrix}=0$.

By (iii), $\delta\begin{pmatrix}1&0\\0&1\end{pmatrix}=1$. Therefore, if we show that $\delta\begin{pmatrix}0&1\\1&0\end{pmatrix}=-1$, we are done.

Using that $(0\ 1)=(1\ 1)-(1\ 0)$,
and $(1\ 0)=(1\ 1)-(0\ 1)$, we have:

$$\delta\begin{pmatrix}0&1\\1&0\end{pmatrix}=\delta\begin{pmatrix}1&1\\1&0\end{pmatrix}-\underbrace{\delta\begin{pmatrix}1&0\\1&0\end{pmatrix}}_{0}=\delta\begin{pmatrix}1&1\\1&1\end{pmatrix}-\delta\begin{pmatrix}1&1\\0&1\end{pmatrix}=$$

$$=-\delta\begin{pmatrix}1&0\\0&1\end{pmatrix}-\underbrace{\delta\begin{pmatrix}0&1\\0&1\end{pmatrix}}_{0}=-\delta(I)=-1.$$

(12) Since $\left|\det\begin{pmatrix}u\\v\end{pmatrix}\right|>0$, we have $\mathcal{O}\begin{pmatrix}u\\v\end{pmatrix}=1\iff\det\begin{pmatrix}u\\v\end{pmatrix}>0$.
Since det is linear in each row,

$\det\begin{pmatrix}u\\v\end{pmatrix}>0\iff\det\begin{pmatrix}\frac{u}{\|u\|}\\\frac{v}{\|v\|}\end{pmatrix}>0$, so we can assume that both u and v have magnitude 1.

$\{u,v\}$ basis $\iff$ v is not a scalar multiple of u $\iff\exists\,\theta\in(0,\pi)\cup(\pi,2\pi)$ s.t. $v=T_\theta(u)$.

Say $u=(a_1,a_2)$. Then $v=T_\theta(u)=$
$=(a_1\cos\theta-a_2\sin\theta,\ a_1\sin\theta+a_2\cos\theta)$.

Thus $\det\begin{pmatrix}u\\v\end{pmatrix}=(a_1^2+a_2^2)\sin\theta>0\iff$
$\iff 0<\theta<\pi\iff\{u,v\}$ is right handed.

HW 10

4.2.4. Choose $a_1 = b_2 = c_3 = 1$ and zero otherwise. then $\det\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = k$

So $k = 2$.

4.2.11

$$\det A = -A_{41}\det\tilde{A}_{41} + A_{42}\det\tilde{A}_{42} - A_{43}\det\tilde{A}_{43} + A_{44}\det\tilde{A}_{44}$$

$$= -3$$

4.2.21 $\det A = A_{11}\det\tilde{A}_{11} - A_{12}\det\tilde{A}_{12} + A_{13}\det\tilde{A}_{13} - A_{14}\det\tilde{A}_{14} = 95$

4.2.22. $\det A = -100$

4.2.23. Prove the statement " the determinant of an upper triangular $n\times n$ matrix is the product of its diagonal entries", denoted by $P(n)$.

Step 1 $P(1)$ is true. this is clear

Step 2 Suppose $P(n)$ is true, it suffices to Prove $P(n+1)$ is true.

Choose arbitrary $A \in M_{(n+1)\times(n+1)}$ upper triangular.

By theorem 4.4. (for $i=n+1$) ~~$\det A = \sum_{j=1}^{n+1} (-1)^{1+n+j} A_{ij} \det \tilde{A}_{ij}$. But~~ A

$$\det A = \sum_{j=1}^{n+1} (-1)^{n+1+j} A_{n+1,j} \det\tilde{A}_{n+1,j} \qquad \text{But } A_{n+1,j} = 0 \text{ if } j \neq n+1.$$

So $\det A = A_{n+1,n+1}\det\tilde{A}_{n+1,n+1}$ ①

We know $\tilde{A}_{n+1,n+1}$ is an upper triangular $n\times n$ matrix, so by induction hypothesis i.e. $P(n)$ we know $\det\tilde{A}_{n+1,n+1} = (\tilde{A}_{n+1,n+1})_{11} \cdots (\tilde{A}_{n+1,n+1})_{nn}$

$$= A_{11}\cdots A_{nn} \qquad ②$$

By ①, ②. $\det A = A_{11}\cdots A_{n+1,n+1}$ So $P(n+1)$ is true.

4.2.25 ~~[illegible]~~

Use Theorem 4.3 repeatedly. write $A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. then $kA = \begin{pmatrix} ka_1 \\ \vdots \\ ka_n \end{pmatrix}$.

then ~~[illegible]~~ $\det(kA) = \det\begin{pmatrix} ka_1 \\ \vdots \\ ka_n \end{pmatrix} = k\det\begin{pmatrix} a_1 \\ ka_2 \\ \vdots \\ ka_n \end{pmatrix} = \cdots = k^n \det\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = k^n \det A$.

4.2.26. $k=-1$ in exe. 25 $\det(-A) = (-1)^n \det A$ so ~~A~~ n should be even.

4.2.30. $A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ $B = \begin{pmatrix} a_n \\ \vdots \\ a_1 \end{pmatrix}$. Notice that $B = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} A$

So it suffices to calculate $\det \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}_{n \times n}$.

Denote the matrix by C_n.

$$\det C_n = \sum_{j=1}^{n} (-1)^{1+j} (C_n)_{1j} \det \widetilde{(C_n)}_{1j} = (-1)^{1+n} \det \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}_{(n-1)\times(n-1)}.$$

$$= (-1)^{n-1} \det C_{n-1}.$$

So $\det C_n = (-1)^{(n-1)+(n-2)+\cdots+1} \det C_1 = (-1)^{\frac{n(n-1)}{2}}$.

Hence $\det B = \det C_n \det A = (-1)^{\frac{n(n-1)}{2}} \det A$.