Section 3.1

(2) $A \rightsquigarrow B$ subtract 2·(first column) from the second column.

$B \rightsquigarrow C$ subtract I row from II row.

$C \rightsquigarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 0 & -3 & -2 \end{bmatrix}$ subtract I row from III row

$\begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 0 & -3 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -3 & -2 \end{bmatrix}$ subtract 3·(I column) from III column.

Get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & -2 \end{bmatrix}$ by subtracting III row from II row.

Get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ by subtracting $\frac{3}{2}$·(III column) from II column.

Get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by multiplying III row by $-\frac{1}{2}$.

(3) (a) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$.

(6) If B is obtained from A by an elementary row operation, we have by Thm. 3.1 that $B = EA$ for some

elementary matrix E. Thus $B^t = A^t E^t$.
It is easy to check that for each type of elementary operation on the rows of I_n with matrix E, the result of the same operations on the columns of I_n has matrix E^t.
Therefore $B^t = A^t E^t$ is obtained from A^t by the corresponding column operation of E.
Similarly for columns.

(9) Let v_i, v_j be two rows of A. Then interchanging them is equivalent to:

- add j-th row to i-th row.
- subtract i-th row from j-th row
- add j-th row to ith row
- multiply j-th row by -1.

Similarly for columns.

(12) Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$.

$\forall j = 2, \ldots, m$, subtract from j-th row:
$\left(-\frac{a_{j1}}{a_{11}}\right) \cdot (\text{I row})$. Then iterate the process:
after k steps, $\forall j = k+1, \ldots, m$ subtract from j-th row: $\left(-\frac{b_{jk}}{b_{kk}}\right) \cdot (k\text{-th row})$

where $B = \left[\begin{array}{ccc|cccc} b_{11} & b_{12} & & * & & * & \\ 0 & b_{22} & & & & & \\ \hline & 0 & 0 & b_{kk} & * & \cdots & * \\ \vdots & & & * & * & \cdots & * \\ 0 & 0 & 0 & * & * & \cdots & * \end{array}\right]$ is the matrix after k steps.

HW8

3.2.2. (a) rank = 2 (b) rank = 3. (c) rank = 2.

(d) rank = 1 (e) rank = 3 (f) rank = 4. (g) rank = 1

3.2.5.

(a) $A^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$ rank $A = 2$ ~~rk = 2~~,

(b) A^{-1} ~~not~~ does not exist rank $A = 1$. ~~rk (A) = 1~~.

(c) A^{-1} does not exist, rank $(A) = 2$

(d) $A^{-1} = \begin{pmatrix} \frac{1}{6} & \frac{15}{9} & -\frac{1}{3} \\ -\frac{5}{18} & -\frac{4}{9} & \frac{2}{9} \\ -\frac{1}{9} & -\frac{2}{9} & \frac{1}{9} \end{pmatrix}$ rank $A = 3$

(e) $A^{-1} = \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$, rank $A = 3$

(f) A^{-1} does not exist rank $A = 2$

(g) $A^{-1} = \begin{pmatrix} -51 & 15 & 7 & 2 \\ 31 & -9 & -4 & -7 \\ -10 & 3 & 1 & 2 \\ -3 & 1 & 1 & 1 \end{pmatrix}$. rank $A = 4$

(h) $A^{-1} = \begin{pmatrix} -3/2 & -\frac{1}{10} & \frac{13}{10} & \frac{1}{10} \\ \frac{1}{2} & \frac{1}{2} & -\frac{2}{5} & \frac{1}{2} \\ -3 & -\frac{1}{5} & \frac{16}{10} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{10} & -\frac{3}{10} & \frac{-1}{10} \end{pmatrix}$. rank $A = 4$.

3.2.7

$$(A, I) = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & & \\ 1 & 0 & 1 & & 1 & \\ 1 & 1 & 2 & & & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & & \\ 0 & -2 & 0 & -1 & 1 & \\ 0 & -1 & 1 & -1 & & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & & \\ & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \\ & -1 & 1 & -1 & & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & & \\ & 1 & & \frac{1}{2} & -\frac{1}{2} & \\ & & 1 & -\frac{3}{2} & \frac{1}{2} & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & & & \frac{3}{2} & \frac{1}{2} & -1 \\ & 1 & & \frac{1}{2} & -\frac{1}{2} & \\ & & 1 & -\frac{3}{2} & \frac{1}{2} & 1 \end{array}\right] \quad \therefore A^{-1} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -1 \\ \frac{1}{2} & -\frac{1}{2} & \end{bmatrix}$$

3.2.7 $\begin{pmatrix} 1 & & \\ 1 & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & \\ 1 & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & -2 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix}$

3.2.12 Because of Thm 3.1, That B' can be transformed into D' by an elementary row ~~colu~~ operation is equivalent to that there exists an $m \times m$ ~~elementary~~ invertible matrix E' s.t. $E'B' = D'$

Put $E = \begin{bmatrix} 1 & \\ & E' \end{bmatrix} = \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & E' & \\ 0 & & & \end{array}\right]$, then $EB = D$.

~~Since E' is elementary, E' is invertible thanks to Thm 3.2~~.

By Corollary 3 to Thm 3.6, ~~the E'~~ the invertibility of E' implies $E' = E_p' E_{p-1}' \cdots E_1'$ where E_i' $(1 \le i \le p)$ are all elementary matrices.

Put $E_i = \begin{bmatrix} 1 & \\ & E_i' \end{bmatrix}$, then clearly E_i is an elementary matrix.

So $E = \begin{bmatrix} 1 & \\ & E' \end{bmatrix} = \begin{bmatrix} 1 & \\ & E_p' \cdots E_1' \end{bmatrix} = \begin{bmatrix} 1 & \\ & E_p' \end{bmatrix} \cdots \begin{bmatrix} 1 & \\ & E_1' \end{bmatrix} = E_p \cdots E_1$ is a product of elementary matrices too.

So $EB = D \Rightarrow E_p \cdots E_1 B = D$, which completes the proof.

3.2.17. (1) By theorem 3.7, $\operatorname{rank}(BC) \le \operatorname{rank}(B) \le 1$

(2). By the Corollary 1 to Thm 3.6, there exist invertible 3×3 matrices P, Q such that $\begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} = PAQ$.

i.e. $A = P^{-1} \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} Q^{-1} = \left[P^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] \cdot \left[(1, 0, 0) \cdot Q^{-1} \right]$

Put $B = P^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, ~~C=~~ $C = (1, 0, 0)\, Q^{-1}$