[redacted]

## Section 2.6

3)

a) $\begin{cases} f_1(1,0,1)=1 \\ f_1(1,2,1)=0 \\ f_1(0,0,1)=0 \end{cases} \Longrightarrow \begin{cases} f_1(e_1)+f_1(e_3)=1 \\ f_1(e_1)+2f_1(e_2)+f_1(e_3)=0 \\ f_1(e_3)=0 \end{cases} \Longrightarrow$

$\Longrightarrow \begin{cases} f_1(e_1)=1 \\ f_1(e_2)=-\frac{1}{2} \\ f_1(e_3)=0 \end{cases} \Longrightarrow f_1(x,y,z) = x-\frac{1}{2}y$

Similarly, $f_2(x,y,z)=\frac{1}{2}y$, $f_3(x,y,z)=z-x$.

(b) As above, we get: $f_1(a_0+a_1x+a_2x^2)=a_0$,

$f_2(a_0+a_1x+a_2x^2)=a_1$, $f_2(a_0+a_1x+a_2x^2)=a_2$.

4) Since $\dim V^* = \dim V = 3$, to show $\{f_1, f_2, f_3\}$ is a basis it is enough to show they are L.I.
If:
$af_1+bf_2+cf_3=0$ for some $a,b,c\in R$,

we get $(a+b)x+(-2a+b+c)y+(b-3c)z=0 \Longrightarrow$

$\Longrightarrow \begin{cases} a+b=0 \\ -2a+b+c=0 \\ b-3c=0 \end{cases} \Longrightarrow a=b=c=0.$

To find a basis for $V$, we consider the inverse process of Exercise 3.
To find the first vector:

$$\begin{cases} x - 2y = 1 \\ x + y + z = 0 \\ y - 3z = 0 \end{cases} \Longrightarrow \begin{cases} x = \frac{2}{5} \\ y = -\frac{3}{10} \\ z = -\frac{1}{10} \end{cases}$$

Similarly for the other vectors, so that the required basis is:

$$\left\{\left(\frac{2}{5}, -\frac{3}{10}, -\frac{1}{10}\right), \left(\frac{3}{5}, \frac{3}{10}, \frac{1}{10}\right), \left(\frac{1}{5}, \frac{1}{10}, -\frac{3}{10}\right)\right\}.$$

7 (a) $T(a_0 + a_1 x) = (-2a_1 - a_0,\ a_0 + a_1)$.

So $T^t f(a_0 + a_1 x) = f T(a_0 + a_1 x) =$
$= -2a_1 - a_0 \ -2(a_0 + a_1) = -4a_1 - 3a_0$.

(b) If $\gamma^* = (f_1, f_2)$, $T^t(f_1)(a_0 + a_1 x) = -a_0 - 2a_1$

and $T^t(f_2)(a_0 + a_1 x) = a_0 + a_1$, thus:

$$[T^t]_{\gamma^*}^{\beta^*} = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$$

(c) Since $\beta = (1, x)$, and $T(1) = (-1, 1)$,
$T(x) = (-2, 1)$, we have:

$$[T]_\beta^\gamma = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}.$$

Thus $\left([T]_\beta^\gamma\right)^t = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} = [T^t]_{\gamma^*}^{\beta^*}$.

(9) ($\Leftarrow$) Assume $T(x) = (f_1(x), \ldots, f_m(x))$.
Then $T(cx+y) = cT(x) + T(y)$ since
$f_i(cx+y) = cf_i(x) + f_i(y)$ $\forall i = 1, \ldots, m$.

($\Rightarrow$) Assume $T$ linear, and let $f_i(x) := T^t(g_i)(x)$
$\forall i = 1, \ldots, m$.
Write $T(x) = (T_1(x), \ldots, T_m(x))$; then clearly

$$T_i(x) = g_i T(x) = T^t(g_i)(x) = f_i(x) \quad \forall i = 1, \ldots, m.$$

I.e., $T(x) = (f_1(x), \ldots, f_m(x))$.

(20) (a) ($\Rightarrow$) Assume $T^t$ not injective. Then
$\exists f \in W^*$ <u>nonzero</u> s.t. $T^t f = 0$.
Now, since $T$ is onto, $R(T) = W$; thus
$f(W) = f(T(V)) = (T^t f)(V) = \{0\}$,
contradiction since $f$ is nonzero.

($\Leftarrow$) Assume $T$ not onto, i.e. $R(T) \subsetneq W$.
Then, by Exercise 19, $\exists f \in W^*$ <u>nonzero</u> s.t.
$f(x) = 0$ $\forall x \in R(T)$.
Thus $(T^t f)(v) = f(T(v)) = 0$ $\forall v \in V$, i.e.
$f \in N(T^t)$.

Therefore $T^t$ is not injective (since $f$ is nonzero), a contradiction.

(b) ($\Leftarrow$) If $T$ is injective, $T: V \xrightarrow{\sim} T(V)$ is isomorphism; thus by extending a basis of $T(V)$ we can find a subspace $Z \subset W$ s.t. $W = T(V) \oplus Z$. Thus we have a projection $p: W \longrightarrow V$. along $Z$.
Now take any $f \in V^*$. Let $g \in W^*$ be defined as $g := fp$.
Then $(T^t g)(x) = (f p T)(v) = f(v) \quad \forall v \in V$.
Thus $T^t$ is surjective.

($\Rightarrow$) Let $v \in V$ s.t. $T(v) = 0$. Assume $v \neq 0$.
Then it's easy to construct $f \in V^*$ s.t. $f(v) \neq 0$. $T^t$ onto $\Rightarrow \exists g \in W^*$ s.t. $T^t g = f$. But then:
$0 \neq f(v) = (T^t g)(v) = g(T(v)) = g(0) = 0$
Contradiction!