Let A, B and C be three matrices, of sizes $m \times n$, $n \times p$ and $p \times q$ respectively. We want to show that $(AB)C = A(BC)$. It is clearly enough to show that all entries are the same, i.e., that $((AB)C)_{ij} = (A(BC))_{ij}$ for any $i = 1, ..., m$ and $j = 1, ..., q$ fixed.

$$((AB)C)_{ij} = \sum_{k=1}^{p} (AB)_{ik} C_{kj} = \sum_{k=1}^{p} (\sum_{h=1}^{n} A_{ih} B_{hk}) C_{kj} = \sum_{k=1}^{p} \sum_{h=1}^{n} A_{ih} B_{hk} C_{kj} =$$

$$= \sum_{h=1}^{n} \sum_{k=1}^{p} A_{ih} B_{hk} C_{kj} = \sum_{h=1}^{n} A_{ih} (\sum_{k=1}^{p} B_{hk} C_{kj}) = \sum_{h=1}^{n} A_{ih} (BC)_{hj} = (A(BC))_{ij}.$$

Section 2.4

Exercise 3

By Theorem 2.19, two finite dimensional vector spaces are isomorphic if and only if they have the same dimension.
a) No, since $\dim(F^3) = 3$ and $\dim(P_3(F)) = 4$.
b) Yes, since $\dim(F^4) = 4$ and $\dim(P_3(F)) = 4$.
c) Yes, since $\dim(M_{2\times 2}(R)) = 4$ and $\dim(P_3(R)) = 4$.
d) No, since $\dim(V) = 3$ (see section 1.6, exercise 16) and $\dim(R^4) = 4$.

Exercise 4

We need to find a matrix M such that $ABM = MAB = I$. Since A and B are invertible, they have inverse matrices A^{-1} and B^{-1}. Let $M := B^{-1}A^{-1}$. Then $ABM = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$, and similarly $MAB = B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$. Therefore AB is invertible, and since the inverse matrix is unique, its inverse has to be $B^{-1}A^{-1}$.

Exercise 10

(a) Since $AB = I$ and I is clearly invertible (with inverse I), AB is invertible (with inverse AB) and by Exercise 9 A and B are also invertible.

(b) Since $AB = I$, if we multiply both sides by B^{-1} on the right, we get $A = ABB^{-1} = IB^{-1} = B^{-1}$. Hence, $B = (B^{-1})^{-1} = A^{-1}$.
(Note that similarly, if we multiply both sides by A^{-1} on the left, we get $B = A^{-1}AB = A^{-1}I = A^{-1}$, but this is done more easily as above.)

(c) Let $T : V \to W$ and $U : W \to Z$ be linear transformations between spaces of the same dimension n. Assume $UT = I$ (the identity map), or more generally that it is an isomorphism. Then U and T are isomorphisms, and $U = T^{-1}$.

To see this, recall that, since UT is an isomorphism, by exercise 12 in section 2.3, T is injective and U is surjective. Since $\dim(V) = \dim(W)$, respectively $\dim(W) = \dim(Z)$, this implies (by Theorem 2.5) that T is an isomoprhism, respectively U is an isomorphism. Since $UT = I$, if we compose both sides on the right with T^{-1}, we get $U = UTT^{-1} = IT^{-1} = T^{-1}$. (Another possible way to prove the statement is to use Figure 2.2 and what we already proved in 10.a and 10.b.)

Exercise 16

By Exercise 10.c, it is enough to find $\Psi : M_{n\times n}(F) \to M_{n\times n}(F)$ such that $\Phi\Psi = I$ (where I is the identity map). Define $\Psi : M_{n\times n}(F) \to M_{n\times n}(F)$ by $\Psi(A) := BAB^{-1}$. Then, for any $A \in M_{n\times n}(F)$,

$$\Phi(\Psi(A)) = \Phi(BAB^{-1}) = B^{-1}BAB^{-1}B = A.$$

I.e., $\Phi\Psi = I$.

Exercise 20

By Theorem 2.21, ϕ_β and ϕ_γ are isomorphisms. In particular, ϕ_β^{-1} is an isomorphism too, and therefore $\phi_\beta^{-1}(F^n) = V$. Since, by Figure 2.2, $L_A\phi_\beta = \phi_\gamma T$, we also have $L_A = \phi_\gamma T\phi_\beta^{-1}$. Applying this linear transformation to F^n, we have that

$$L_A(F^n) = \phi_\gamma T\phi_\beta^{-1}(F^n) = \phi_\gamma(T(V)).$$

Therefore $\dim(L_A(F^n))$ $=\dim(\phi_\gamma(T(V)))$. Now, by Exercise 17.b, $\dim(\phi_\gamma(T(V)))$ $=\dim(T(V)))$. Therefore $\dim(L_A(F^n))$ $=\dim(T(V)))$, i.e. $\text{rank}(T) = \text{rank}(L_A)$.

Since $\text{rank}(T) + \text{nullity}(T) = n = \text{rank}(L_A) + \text{nullity}(L_A)$ by the Dimension Theorem, and $\text{rank}(T) = \text{rank}(L_A)$, we clearly also have that $\text{nullity}(T) = \text{nullity}(L_A)$.

HW6

2.5.4. • $Q = [I_V]_{\beta}^{\beta'}$

Put $\beta = \left\{ \binom{1}{0}, \binom{0}{1} \right\} = \{e_1, e_2\}$, $\beta' = \left\{ \binom{1}{1}, \binom{1}{2} \right\} = \{f_1, f_2\}$.

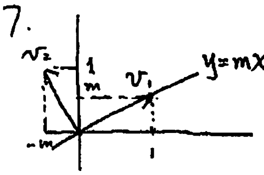
$\begin{cases} f_1 = e_1 + e_2 \\ f_2 = e_1 + 2e_2 \end{cases} \Rightarrow Q = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

• $\begin{cases} Te_1 = 2e_1 + e_2 \\ Te_2 = e_1 - 3e_2 \end{cases} \Rightarrow [T]_\beta = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$

• By Thm 2.23 $[T]_{\beta'} = Q^{-1}[T]_\beta Q = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 13 \\ -5 & 9 \end{bmatrix}$.

2.5.6 (a) $\begin{pmatrix} 6 & 11 \\ -2 & 4 \end{pmatrix}$, (b) $\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$, (c) $\begin{pmatrix} 2 & 2 & 2 \\ -2 & -3 & -4 \\ 1 & 1 & 2 \end{pmatrix}$, (d) $\cancel{-24\begin{pmatrix} 6 & 0 & 5 \\ 0 & 18 & 0 \\ 0 & 6 & 31 \end{pmatrix}}$ $\begin{pmatrix} 6 & & \\ & 12 & \\ & & 18 \end{pmatrix}$

2.5.7.



(a) Consider $\beta' = \{v_1, v_2\} = \left\{ \binom{1}{m}, \binom{-m}{1} \right\}$.

Since T is a reflection about L, we must have $Tv_1 = v_1$ and $Tv_2 = -v_2$

So $[T]_{\beta'} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Let $\beta = \{e_1, e_2\} = \left\{ \binom{1}{0}, \binom{0}{1} \right\}$.

~~(b) $T(av_1 + bv_2) = av_1$~~ Then $Q = [I_V]_{\beta'}^{\beta} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}$

because $\begin{cases} v_1 = e_1 + me_2 \\ v_2 = -me_1 + e_2 \end{cases} \Rightarrow (v_1, v_2) = (e_1, e_2) \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}$.

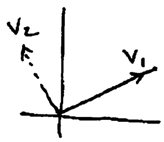
From $[T]_{\beta'} = Q^{-1}[T]_\beta Q$, we know

$[T]_\beta = Q[T]_{\beta'}Q^{-1} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} \frac{1}{1+m^2}$

$= \begin{pmatrix} 1 & m \\ m & -1 \end{pmatrix} \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} \frac{1}{1+m^2} = \begin{pmatrix} \frac{1-m^2}{1+m^2}, & 2m \\ 2m, & \frac{m^2-1}{1+m^2} \end{pmatrix}$

HW 6

2.5.7 (b). $\beta' = \{v_1, v_2\} = \left\{ \begin{pmatrix} 1 \\ m \end{pmatrix}, \begin{pmatrix} -m \\ 1 \end{pmatrix} \right\}$, then $Tv_1 = v_1$, $Tv_2 = 0$



$\Rightarrow [T]_{\beta'} = \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}$.

Similar to (a),

$$[T]_\beta = Q\,[T]_{\beta'}\,Q^{-1} = \begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} \frac{1}{1+m^2}$$

$$= \begin{pmatrix} 1 & 0 \\ m & 0 \end{pmatrix} \begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix} \frac{1}{1+m^2} = \begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix} \frac{1}{1+m^2}$$

2.5.8. $\cancel{\gamma' [T]_{\beta'}^{\gamma'} = T\beta' =}$

$$\begin{aligned}
(\gamma_1', \cdots, \gamma_n')\,[T]_{\beta'}^{\gamma'} &= T(\beta_1', \cdots, \beta_m') = T\,I_V\,(\beta_1' \cdots, \beta_m') \\
&= T(\beta_1, \cdots, \beta_m)\,[I_V]_{\beta'}^{\beta} \\
&= (\gamma_1 \cdots \gamma_n)\,[T]_\beta^\gamma\,[I_V]_{\beta'}^{\beta} \\
&= I_W(\gamma_1, \cdots \gamma_n)\,[T]_\beta^\gamma\,[I_V]_{\beta'}^{\beta} \\
&= (\gamma_1' \cdots \gamma_n')\,[I_W]_\gamma^{\gamma'}\,[T]_\beta^\gamma\,[I_V]_{\beta'}^{\beta}
\end{aligned}$$

So $[T]_{\beta'}^{\gamma'} = P^{-1}\,[T]_\beta^\gamma\,Q$ where $Q = [I_V]_{\beta'}^{\beta}$, $P = [I_W]_{\gamma'}^{\gamma}$

2.5.10. $\exists\, Q$ invertible s.t. $A = Q^{-1} B Q$

then $tr(A) = tr(Q^{-1}(BQ)) = tr((BQ)Q^{-1}) = tr(B(QQ^{-1})) = tr(B)$

2.5.11

(a) $Q = [I_V]_\alpha^\beta$, $R = [I_V]_\beta^\gamma$. $(\gamma_1 \cdots \gamma_n)[I_V]_\alpha^\gamma = I_V(\alpha_1 \cdots \alpha_n)$

$= (\beta_1 \cdots \beta_n)\,[I_V]_\alpha^\beta = I_V(\beta_1 \cdots \beta_n)\,[I_V]_\alpha^\beta$

$~~RQ = [I_V]_\beta^\gamma [I_V]_\alpha^\beta = [I_V]_\alpha^\gamma~~$

$= (\gamma_1 \cdots \gamma_n)\,[I_V]_\beta^\gamma\,[I_V]_\alpha^\beta$

~~(b) On the other~~ $\Rightarrow [I_V]_\alpha^\gamma = [I_V]_\beta^\gamma\,[I_V]_\alpha^\beta$ i.e. $RQ = [I_V]_\alpha^\gamma$

2.5.11 (b) $\because (\beta_1, \cdots, \beta_n) = (\alpha_1 \cdots \alpha_n) [I_V]^{\alpha}_{\beta} = (\beta_1 \cdots \beta_n) [I_V]^{\beta}_{\alpha} [I_V]^{\alpha}_{\beta}$

$\therefore [I_V]^{\beta}_{\alpha} [I_V]^{\alpha}_{\beta} = I$ Similarly. $[I_V]^{\alpha}_{\beta} [I_V]^{\beta}_{\alpha} = I$

$\therefore Q^{-1} = [I_V]^{\alpha}_{\beta}$

2.5.13 $$(x_1', \cdots, x_n') = (x_1 \cdots x_n) \begin{bmatrix} Q_{11} & \cdots & Q_{1n} \\ \vdots & & \vdots \\ Q_{n1} & \cdots & Q_{nn} \end{bmatrix}$$ ~~i.e. $\beta' = \beta$~~

$\therefore \cancel{[Q]^{\beta} Q = [I_V]^{\beta'}_{\beta'}}$ $\therefore Q = [I_V]^{\beta}_{\beta'}$

2.5.14. Put $V = F^n$, $W = F^m$, $T = L_A$.

Let $\beta = \{\beta_1 \cdots \beta_n\}$, $\gamma = \{\gamma_1 \cdots \gamma_m\}$ be standarm ordered bases for F^n and F^m respectively.

Then it is easy to show $[T]^{\gamma}_{\beta} = A$.

Let $\beta' = \{\beta_1' \cdots, \beta_n'\}$ $\gamma' = \{\gamma_1' \cdots \gamma_m'\}$ s.t. $Q = (\beta_1' \cdots, \beta_n')$ (each β_i' is a column) and $P = (\gamma_1', \cdots, \gamma_m')$.

Then $B = P^{-1} A Q$ implies $L_A (\beta_1' \cdots, \beta_n') = (\gamma_1' \cdots \gamma_m') B$.

that is $B = [L_A]^{\gamma'}_{\beta'} = [T]^{\gamma'}_{\beta'}$