2.2.2. ~~Let $e_1 = (1, 0 \cdots 0), \cdots, e_n = (0, \cdots 0, 1)$~~.

(a) $\begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ (c) $[2, 1, -3]$

(d) $\begin{bmatrix} 0 & 2 & 1 \\ -1 & 4 & 5 \\ 1 & 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & & & \\ 1 & 0 & \cdots & 0 \end{bmatrix}_{n\times n}$ (f) $\begin{bmatrix} & & & 1 \\ & & \iddots & \\ & 1 & & \\ 1 & & & \end{bmatrix}_{n\times n}$

(g) $[1, 0 \cdots 0, 1]_{1\times n}$.

2.2.5.

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}$

(d) $[1, 2, 4]$ ~~(e)~~ $\cancel{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & \\ 0 & 0 & \end{bmatrix}} \cancel{\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}} \cancel{\begin{bmatrix} 1 \\ -2 \\ 0 \\ 4 \end{bmatrix}}$

~~(f)~~ $\cancel{\begin{bmatrix} 2 & 3 \\ 0 & -6 \\ 0 & 1 \end{bmatrix}}$

(e) $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 4 \end{bmatrix}$ (f) $\begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}$ (g) $[a]$.

2.1.25 (b) $T(a,b,c) = (0,0,c)$

(c) • $\underline{W_1 \oplus L = R^3}$

(i) $W_1 \cap L = \{0\}$? : $(a,b,c) \in W_1 \cap L \Rightarrow (a,b,c) \in W_1$ and $(a,b,c) \in L$

$\Rightarrow c = 0$ and $b = 0$, $a = c$

$\Rightarrow a = b = c = 0$

(ii) $W_1 + L = R^3$? :

$(a,b,c) = (a-c, b) + (c, 0, c)$ where $(a-c, b, 0) \in W_1$ and $(c, 0, c) \in W_2$.

Now, just according to the definition, $T(a,b,c) = (a-c, b, 0)$ is a projection along L

2.1.27

(a) • W is a subspace of V, so has a basis, say $S_1 = \{w_1, \cdots, w_k\}$. (By the corrollary of Thm 1.13)

By Thm 1.13, $\exists$ a maximal linearly independent subset $S_2 = \{w_1 \cdots w_k, w_{k+1} \cdots w_n\}$ of V that contains $S_1 = \{w_1, \cdots, w_k\}$.

Now, according to Thm 1.12 (we just take $S = V$ in this theorem), S_2 is a basis for V.

• We define $W' = \mathrm{Span}\{w_{k+1}, \cdots, w_n\}$, then $W \cap W' = \{0\}$ since S_2 is linearly independent. and $W + W' = V$ since S_2 generates V. So $V = W \oplus W'$

• We define $T: V \to V$ by $a_1 w_1 + \cdots + a_n w_n \mapsto a_1 w_1 + \cdots + a_k w_k$, and T is a projection on W along W'.

(b) We use the same notation in (a) and assume $k = \dim W < \dim V = n$, and define $W'' = \mathrm{Span}\{w_{k+1} + w_{k+2}, w_{k+2}, \cdots, w_n\}$.

Alternatively, by Corollary 2 to Theorem 1.10, we can also extend $S_1 = \{w_1 \cdots w_k\}$ to a basis $S_2 = \{w_1 \cdots w_n\}$.

2.2.10. $T v_1 = v_1$

$T v_2 = v_1 + v_2$

$\vdots$

$T v_n = v_{n-1} + v_n$

$$\Longrightarrow T(v_1, \cdots, v_n) = (v_1, \cdots, v_n)\begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & \ddots & \\ & & & \ddots & 1 \\ & & & & 1 \end{bmatrix}$$

So $[T]_\beta = \begin{bmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & \ddots \\ & & & \ddots & 1 \\ & & & & 1 \end{bmatrix}$ $\left(\begin{array}{l} \text{e.g.} \\ \text{If } n=3 \quad [T]_\beta = \begin{bmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{bmatrix} \end{array}\right)$.

2.2.12.

Find a basis $\{w_1, \cdots, w_k\}$ for W

Find a basis $\{w_{k+1}, \cdots, w_n\}$ for W'. then ~~for~~ $\beta = \{w_1, \cdots, w_n\}$ is a basis for V

And $Tw_1 = w_1 \cdots Tw_k = w_k$

$Tw_{k+1} = \cdots = T_n = 0$.

So $T(w_1 \cdots w_n) = (w_1 \cdots w_n)\begin{bmatrix} \overbrace{1 & & }^{k} & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}$

$[T]_\beta = \begin{bmatrix} \overbrace{1 & & }^{k} & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}$ is diagonal.

MAT 310
HW 5

Section 2.3

Exercise 2

a) $A(2B+3C) = \begin{bmatrix} 20 & -9 & 18 \\ 5 & 10 & 8 \end{bmatrix}$

$A(BD) = (AB)D = \begin{bmatrix} 29 \\ -26 \end{bmatrix}$

b) $A^t = \begin{bmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{bmatrix}$

$A^tB = \begin{bmatrix} 23 & 19 & 0 \\ 26 & -1 & 10 \end{bmatrix}$

$BC^t = \begin{bmatrix} 12 \\ 16 \\ 29 \end{bmatrix}$

$CB = \begin{bmatrix} 27 & 7 & 9 \end{bmatrix}$

$CA = \begin{bmatrix} 20 & 26 \end{bmatrix}$

Exercise 12

(a) Let $v \in V$ such that $T(v) = 0$. Applying U to both sides, we have $UT(v) = U(T(v)) = U(0) = 0$. Since UT is one-to-one by hypothesis, we get that $v = 0$. Therefore T is one-to-one.

However, U must not be one-to-one. For example, let $T : R^2 \to R^3$ be the linear map $T(a_1, a_2) = (a_1, a_2, 0)$, and let $U : R^3 \to R^3$ be the linear map $U(b_1, b_2, b_3) = (b_1, b_2, 0)$. Then it's easy to check that UT is one-to-one and that $N(U) = span(e_3) \neq \{0\}$.

(b) Let $z \in Z$ be any vector. Since UT is onto by hypothesis, there exists $v \in V$ such that $UT(v) = z$. Since $UT(v) = U(T(v))$, we have that $T(v) \in W$ is a vector whose image under U is z, i.e., U is onto.

Hoever, T must not be onto. For example, let $T : R^2 \to R^2$ be the linear map $T(a_1, a_2) = (a_1, 0)$, and let $U : R^2 \to R$ be the linear map $U(b_1, b_2) = b_1$. Then it's easy to check that UT is onto and that $R(T) = span(e_1)$ (which is a proper subspace of R^2).

(c) Assume that U and T are isomorphisms. Then $UT(v) = 0$ implies (since U is injective) $T(v) = 0$, which implies (since T is injective) $v = 0$. Thus UT is injective. To prove surjectivity, consider any $z \in Z$. Since U is onto, there is some $w \in W$ such that $U(w) = z$. Since T is onto, there is some $v \in V$ such that $T(v) = w$. Therefore $UT(v) = U(w) = z$, i.e. UT is onto.

Exercise 13

$$\mathrm{tr}(AB) = \sum_{i=1}^{n}(AB)_{ii} = \sum_{i=1}^{n}(\sum_{k=1}^{n} A_{ik}B_{ki}) = \sum_{i=1}^{n}(\sum_{k=1}^{n} B_{ki}A_{ik}) = \sum_{k=1}^{n}(\sum_{i=1}^{n} B_{ki}A_{ik}) = \sum_{k=1}^{n}(BA)_{kk} = \mathrm{tr}(BA).$$

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii} = \sum_{i=1}^{n} A^t_{ii} = \mathrm{tr}(A^t).$$

Exercise 15

Say A is a $n \times q$ matrix and M is a $p \times n$ matrix. We are assuming that the j-th column of A, $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}$, is equal to the linear combination

$$\sum_{h \neq j} c_h \begin{pmatrix} a_{1h} \\ a_{2h} \\ \vdots \\ a_{nh} \end{pmatrix} = \begin{pmatrix} \sum_{h\neq j} c_h a_{1h} \\ \sum_{h\neq j} c_h a_{2h} \\ \vdots \\ \sum_{h\neq j} c_h a_{nh} \end{pmatrix}.$$

Then the j-th column of MA is

$$\begin{pmatrix} \sum_k m_{1k}a_{kj} \\ \sum_k m_{2k}a_{kj} \\ \vdots \\ \sum_k m_{pk}a_{kj} \end{pmatrix} = \begin{pmatrix} \sum_k m_{1k}(\sum_{h\neq j} c_h a_{kh}) \\ \sum_k m_{2k}(\sum_{h\neq j} c_h a_{kh}) \\ \vdots \\ \sum_k m_{pk}(\sum_{h\neq j} c_h a_{kh}) \end{pmatrix} = \begin{pmatrix} \sum_{h\neq j} c_h(\sum_k m_{1k}a_{kh}) \\ \sum_{h\neq j} c_h(\sum_k m_{2k}a_{kh}) \\ \vdots \\ \sum_{h\neq j} c_h(\sum_k m_{pk}a_{kh}) \end{pmatrix} = \sum_{h\neq j} c_h \begin{pmatrix} \sum_k m_{1k}a_{kh} \\ \sum_k m_{2k}a_{kh} \\ \vdots \\ \sum_k m_{pk}a_{kh} \end{pmatrix}$$

where $\begin{pmatrix} \sum_k m_{1k}a_{kh} \\ \sum_k m_{2k}a_{kh} \\ \vdots \\ \sum_k m_{pk}a_{kh} \end{pmatrix}$ is exactly the h-th column of MA for every $h \neq j$.

Exercise 18

Let A, B and C be three matrices, of sizes $m \times n$, $n \times p$ and $p \times q$ respectively. We want to show that $(AB)C = A(BC)$. It is clearly enough to show that all entries are the same, i.e., that $((AB)C)_{ij} = (A(BC))_{ij}$ for any $i = 1, ..., m$ and $j = 1, ..., q$ fixed.

$$((AB)C)_{ij} = \sum_{k=1}^{p} (AB)_{ik} C_{kj} = \sum_{k=1}^{p} (\sum_{h=1}^{n} A_{ih} B_{hk}) C_{kj} = \sum_{k=1}^{p} \sum_{h=1}^{n} A_{ih} B_{hk} C_{kj} =$$

$$= \sum_{h=1}^{n} \sum_{k=1}^{p} A_{ih} B_{hk} C_{kj} = \sum_{h=1}^{n} A_{ih} (\sum_{k=1}^{p} B_{hk} C_{kj}) = \sum_{h=1}^{n} A_{ih} (BC)_{hj} = (A(BC))_{ij}.$$