If the linear combiation is zero, by the identity principle for polynomials we must have in particular that $a_n = 0$, a contradiction. Therefore S is linearly independent.

Section 1.6

Exercise 1

a) False, $\{0\}$ is a basis.
b) True, by Theorem 1.9.
c) False. For example, $P(F)$.
d) False. A trivial example is given by $\{1\}$ and $\{2\}$ which are two bases for R.
e) True, by Corollary 1.
f) False, it's $n+1$.
g) False, it's mn.
h) True, by Corollary to Theorem 1.11.
i) False, since S might not be a basis.
j) True, by Theorem 1.11.
k) True, by Theorem 1.11, and because the only subspace of dimension 0 is $\{0\}$.
l) True, by Theorem 1.11.

Exercise 8

$$\begin{aligned} W &= \{(a_1, a_2, a_3, a_4, a_5) \in R^5 \ : \ a_1 + a_2 + a_3 + a_4 + a_5 = 0\} = \\ &= 1\{(a_1, a_2, a_3, a_4, -a_1 - a_2 - a_3 - a_4) \in R^5 \ : \ a_1, a_2, a_3, a_4 \in R\}. \end{aligned}$$

It follows easily that $\dim(W) = 4$ (for example, it's easy to write a basis for W, like in Exercise 9 below). Since $\{u_1, ..., u_8\}$ spans W, and $\dim(W) = 4$, we just need to find 4 vectors in the set $\{u_1, ..., u_8\}$ which are linearly independent. By easy but lengthy computations (using the method of Theorem 1.9 or Example 6), $\{u_1, u_3, u_5, u_7\}$ is such a set.

Exercise 9

We need to write $(a_1, a_2, a_3, a_4) = b_1u_1 + b_2u_2 + b_3u_3 + b_4u_4$ for some b_i, $i = 1, 2, 3, 4$. This amounts to solving the linear system:

$$\begin{cases} a_1 = b_1 \\ a_2 = b_1 + b_2 \\ a_3 = b_1 + b_2 + b_3 \\ a_4 = b_1 + b_2 + b_3 + b_4 \end{cases}$$

It is easy to see that it has unique solution:

$$\begin{cases} b_1 = a_1 \\ b_2 = a_2 - a_1 \\ b_3 = a_3 - a_2 \\ b_4 = a_4 - a_3 \end{cases}.$$

Exercise 14

$$W_1 = \{(a_1, a_2, a_3, a_4, a_5) \in F^5 \ : \ a_1 - a_3 - a_4 = 0\} =$$
$$= \{(a_3 + a_4, a_2, a_3, a_4, a_5) \in F^5 \ : \ a_2, a_3, a_4, a_5 \in F\}.$$

Therefore every vector in W_1 can be written as a linear combination of:

$$v_1 := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 := \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_4 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

It can be easily checked that v_1, v_2, v_3, v_4 are linearly independent. Therefore they form a basis for W_1, and in particular $\dim(W_1) = 4$.

Similarly, $W_2 = \{(a_1, a_2, a_2, a_2, -a_1) \in F^5 \ : \ a_1, a_2 \in F\}$ is spanned by:

$$v_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad v_2 := \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix},$$

which are easily seen to be linearly independent. Thus $\{v_1, v_2\}$ is a basis for W_2, and in particular $\dim(W_2) = 2$.

Exercise 15

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$, $\mathrm{tr}(A) = a_{11} + a_{22} + \dots + a_{nn}$. Therefore:

$$W = \{A \in M_{n \times n}(F) \ : \ a_{nn} = -\sum_{i=1}^{n-1} a_{i}\},$$

i.e., matrices in W are of the form: $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & -a_{11} - \ldots - a_{nn} \end{bmatrix}$.

This means that every matrix $A \in W$ can be written as a linear combination:

$$A = \sum_{i \neq j} a_{ij} E^{ij} + \sum_{k=1}^{n-1} a_{kk} G^k,$$

where $G^k \in M_{n \times n}(F)$ is the matrix given by:

$$(G^k)_{lm} = \begin{cases} 1 & \text{if } l = m = k, \\ -1 & \text{if } l = m = n, \\ 0 & \text{otherwise} \end{cases}.$$

It is easy to check that the set $S := \{E^{ij} \, : \, i, j = 1, ..., n, i \neq j\} \cup \{G^k\}_{k=1,...,n-1}$ is linearly independent. Therefore S is a basis of W. Since S contains $(n^2 - n) + (n - 1) = n^2 - 1$ elements, we have that $\dim(W) = n^2 - 1$.

ANWER4

2.1.5 Definition in page 65

- T is a linear transformation:

(a) $T(f(x)+g(x)) = x(f(x)+g(x)) + (f(x)+g(x))'$

$= (xf(x)+f'(x)) + (xg(x)+g'(x)) = T(f(x)) + T(g(x))$

(b) $T(cf(x)) = cT(f(x))$

- Bases for $N(T)$

Let $f(x) = a_0 + a_1x + a_2x^2 \in P_2(R)$.

$T(f(x)) = a_0x + a_1x^2 + a_2x^3 + (a_1 + 2a_2x) = a_1 + (a_0+2a_2)x + a_1x^2 + a_2x^3$

So, $T(f(x)) = 0$ iff $\begin{cases} a_1 = 0 \\ a_0 + 2a_2 = 0 \\ a_1 = 0 \\ a_2 = 0 \end{cases}$ iff $a_0 = a_1 = a_2 = 0$. iff $f(x) = 0$.

So $N(T) = \{0\}$. (*)

- Bases for $R(T)$

Since $N(T) = \{0\}$, by Theorem 2.3 (Dimension Theorem), we know $\text{rank}(T) = \dim P_3(R)$ ----- (**)

Since $R(T) \subseteq P_3(R)$, by Theorem 1.11, $R(T) = P_3(R)$.

So a base of $R(T) = P_3(R)$ is just $\{1, x, x^2, x^3\}$.

- One-to-one : By Thm 2.4. and (*)
- Onto : By Thm 2.5 and (**)

~~2.1.11~~

- $\{(1,0), (1,1)\}$ is a basis for R^2 Clearly
- By Thm 2.6, $\exists\, T: R^2 \to R^3$ s.t. $T(1,0) = (1,0,2)$, $T(1,1) = (1,-1,4)$
- $(2,3) = -(1,0) + 3(1,1)$

So $T(2,3) = -T(1,0) + 3T(1,1) = \cancel{-(1,4) + 3(2,5) = (5,11)}$

2.1.11. • $S = \{(1,1),(2,3)\}$ is a basis for R^2

Because S is linearly independent and span $S = R^2$

• By Thm 2.6, $\exists\, T: R^2 \to R^3$ s.t. $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$

• $(8,11) = 2*(1,1) + 3(2,3)$

So $T(8,11) = 2T(1,1) + 3T(2,3) = 2(1,0,2) + 3(1,-1,4)$

$= (5,-3,16)$

2.1.13. Suppose $a_1 v_1 + \cdots + a_k v_k = 0$ for some $a_i \in F$. It suffices to prove $a_i = 0$.

In fact, $0 = T(a_1 v_1 + \cdots + a_k v_k)$

$= a_1 T v_1 + \cdots + a_k T v_k = a_1 w_1 + \cdots + a_k w_k$

But $\{w_1, \cdots, w_k\}$ is linearly independent, so $a_1 = \cdots = a_k = 0$

2.1.17. Recall Dimension Theorem:

$(*)$: $\dim(V) = \text{nullity}(T) + \text{rank}(T)$

(a) $\dim V < \dim W \overset{(*)}{\Longrightarrow} \text{nullity}(T) + \text{rank}(T) < \dim W$

$\Rightarrow \text{rank}(T) < \dim W$

$\Rightarrow$ T cannot be onto, according to Thm 2.5.

(b) $\dim(V) > \dim(W) \overset{(*)}{\Longrightarrow} \text{nullity}(T) + \text{rank}(T) > \dim W$

$\Rightarrow \text{nullity}(T) > \dim W - \text{rank}(T) \geq 0$ (Because $R(T) \subseteq W \overset{\text{Thm 1.11}}{\Longrightarrow} \text{rank}(T) \leq \dim W$)

$\Rightarrow$ T cannot be one-to-one, according to Thm 2.4.

2.1.25

(a) Define $W_1 = \{(a,b,0) \mid a,b \in R\}$; $W_2 = \{(0,0,c) \mid c \in R\}$.

• W_1, W_2 are subspaces: Clearly, easy to show.

• $W_1 \oplus W_2 = R^3$:

(cf. Definition in Page 22)

(i) $W_1 \cap W_2 = \{0\}$? $x = (a,b,c) \in W_1 \cap W_2 \Rightarrow x \in W_1$ and $x \in W_2$

$\Rightarrow c = 0$ and $a = b = 0$

(ii) $W_1 + W_2 = R^3$?

$\forall x = (a,b,c) \in R^3$, $x = (a,b,0) + (0,0,c)$

where $(a,b,0) \in W_1$ and $(0,0,c) \in W_2$.

• Now $(a,b,c) = (a,b,0) + (0,0,c)$

and $(a,b,0) \in W_1$

Just according to the definition in Page 76, $T(a,b,c) = (a,b,0)$ is a projection.

2.1.25 (b) $T(a,b,c) = (0,0,c)$

(c) • $\underline{W_1 \oplus L = R^3}$

(i) $W_1 \cap L = \{0\}$? : $(a,b,c) \in W_1 \cap L \Rightarrow (a,b,c) \in W_1$ and $(a,b,c) \in L$

$\Rightarrow c=0$ and $b=0$, $a=c$

$\Rightarrow a=b=c=0$

(ii) $W_1 + L = R^3$? :

$(a,b,c) = (a-c,b) + (c,0,c)$ where $(a-c,b,0) \in W_1$ and $(c,0,c) \in W_2$.

Now, just according to the definition, $T(a,b,c) = (a-c,b,0)$ is a projection along L

2.1.27

(a) • W is a subspace of V, so has a basis, say $S_1 = \{w_1, \cdots, w_k\}$. (By the corrollary of Thm 1.13)

By Thm 1.13, $\exists$ a maximal linearly independent subset $S_2 = \{w_1 \cdots w_k, w_{k+1} \cdots w_n\}$ of V that contains $S_1 = \{w_1, \ldots, w_k\}$.

Now, according to Thm 1.12 (we just take $S = V$ in this theorem), S_2 is a basis for V.

• We define $W' = \mathrm{Span}\{w_{k+1}, \cdots, w_n\}$, then $W \cap W' = \{0\}$ since S_2 is linearly independent. and $W + W' = V$ since S_2 generates V. So $V = W \oplus W'$

• We define $T: V \to V$ by $a_1w_1 + \cdots + a_nw_n \mapsto a_1w_1 + \cdots + a_kw_k$, and T is a projection on W along W'.

(b) We use the same notation in (a) and assume $k = \dim W < \dim V = n$, and define $W'' = \mathrm{Span}\{w_{k+1} + w_{k+2}, w_{k+2}, \cdots, w_n\}$.

Alternatively, by Corollary 2 to Theorem 1.10, we can also extend $S_1 = \{w_1 \cdots w_k\}$ to a basis $S_2 = \{w_1 \cdots w_n\}$.