1.4.2

(a) $\left[\begin{array}{cccc|c} 2 & -2 & -3 & 0 & -2 \\ 3 & -3 & -2 & 5 & 7 \\ 1 & -1 & -2 & -1 & -3 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_3} \left[\begin{array}{cccc|c} 1 & -1 & -2 & -1 & -3 \\ 3 & -3 & -2 & 5 & 7 \\ 2 & -2 & -3 & 0 & -2 \end{array}\right]$

$\xrightarrow[r_3 - 2r_1]{r_2 - 3r_1} \left[\begin{array}{cccc|c} 1 & -1 & -2 & -1 & -3 \\ 0 & 0 & 4 & 8 & 16 \\ 0 & 0 & 1 & 2 & 4 \end{array}\right] \xrightarrow{r_2 - 4r_3} \left[\begin{array}{cccc|c} 1 & -1 & -2 & -1 & -3 \\ & & 0 & 0 & 0 \\ & & 1 & 2 & 4 \end{array}\right]$

i.e. $\xrightarrow{r_1 + 2r_3} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 3 & 5 \\ & & 1 & 2 & 4 \end{array}\right]$ i.e. $\begin{cases} x_1 - x_2 + 3x_4 = 5 \\ x_3 + 2x_4 = 4. \end{cases}$

Put $x_2 = s$, $x_4 = t$, then we have

$x_1 = 5 + s - 3t$, $x_2 = s$, $x_3 = 4 - 2t$, $x_4 = t$.

(b) $x_1 = -2$, $x_2 = -4$, $x_3 = -3$

(c) No solution

(d) ~~No solution~~ $x_1 = -8t - 16$, $x_2 = 3t + 9$, $x_3 = t$, $x_4 = 2$

(e) ~~No solution~~ $x_1 = 10r - 3s - 4$, $x_2 = -3r + 2s + 3$, $x_3 = r$, $x_4 = s$, $x_5 = 5$.

(f) $x_1 = 3$, $x_2 = 4$, $x_3 = -2$

1.4.4

(a) $x^3 - 3x + 5 = a(x^3 + 2x^2 - x + 1) + b(x^3 + 3x^2 - 1)$

$\Leftrightarrow \begin{cases} 1 = a + b \\ 0 = 2a + 3b \\ -3 = -a \\ 5 = a - b \end{cases} \Leftrightarrow \begin{cases} a = 3 \\ b = -2 \end{cases}$

(b) No. (c). $\begin{cases} a = 4 \\ b = -3 \end{cases}$ (d) $\begin{cases} a = -2 \\ b = 5 \end{cases}$ (e) No.

(f) No.

1.4.5. (Similar argument to 1.4.4)

(a) Yes (b) No (c) No (d) Yes.

(e) Yes (f) No. (g) Yes (h) No

1.4.7. For any $x = (a_1, \cdots, a_n) \in F^n$

$x = a_1 e_1 + \cdots + a_n e_n \in \mathrm{Span}\{e_1, \cdots, e_n\}$

So $F^n \subseteq \mathrm{Span}\{e_1 \cdots e_n\}$.

So $F^n = \mathrm{Span}\{e_1, \ldots e_n\}$.

1.4.12 <u>"if" part</u>: (a) $0 \in \mathrm{Span}\, W = W$

(b) If $x, y \in W$, then $x+y \in \mathrm{Span}\, W = W$

(c) If $c \in F$, $x \in W$, ~~ex~~ then $cx \in \mathrm{span}\, W = W$

So, by Thm 1.2, $W$ is a subspace.

<u>"only if" part</u>: clearly $W \subseteq \mathrm{span}\, W$.

Let $x \in \mathrm{span}\, W$, then $x$ is of form $x = a_1 w_1 + \cdots + a_k w_k$

where $a_i \in F$ and $w_i \in W$.

But since $W$ is a subspace, by exercise 1.3.20, $x \in W$.

So $\mathrm{span}\, W \subseteq W$

1.4.15. (1) $S_1 \cap S_2 \subseteq S_1$ implies $\mathrm{Span}(S_1 \cap S_2) \subseteq \mathrm{Span}(S_1)$

Similarly $\mathrm{Span}(S_1 \cap S_2) \subseteq \mathrm{Span}(S_2)$

So $\mathrm{Span}(S_1 \cap S_2) \subseteq \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$.

(2) ① Let $v \neq w$ are two vectors in $V$.

Put $S_1 = \{v, v+w\}$ $S_2 = \{-v+w, w\}$,

Then $\mathrm{Span}\, S_1 = \mathrm{Span}\{v, w\}$, $\mathrm{Span}\, S_2 = \mathrm{Span}\{v, w\}$

But $S_1 \cap S_2 = \emptyset$.

② Put $S_1 = \{v, w\}$, $S_2 = \{w\}$.

1.4.16. Suppose $x \in S$ has two different ways to write as a linear combination of vectors of $S$.

Say ~~$x = a_1 v_1 + \cdots + a_n v_n = b_1 w_1 + \cdots + b_m w_m$~~

~~where~~ $x = a_1 v_1 + \cdots + a_k v_k + b_1 u_1 + \cdots + b_\ell u_\ell$

$= a_1' v_1 + \cdots + a_k' v_k + c_1 w_1 + \cdots + c_m w_m$

where $v_1 \cdots v_k, u_1 \cdots u_\ell, w_1 \cdots w_m$ are distinct vectors in $S$.

and $a_i, b_i, c_i, a_i' \in F$.

Then $(a_1 - a_1') v_1 + \cdots + (a_k - a_k') v_k + b_1 u_1 + \cdots + b_\ell u_\ell + (-c_1) w_1 + \cdots + (-c_m) w_m = 0$

By the condition, $a_1 - a_1' = \cdots = a_k - a_k' = 0$ $b_i = c_j = 0$.

# MAT 310
# HW 3

## Section 1.5

### Exercise 1

a) False, possibly just one.
b) True, since $\{0\}$ is already linearly dependent.
c) False, by definition.
d) False. For example, $\{e_1, e_2\} \subset \{e_1, e_2, e_1 + e_2\} \subset R^2$.
e) True, by Corollary to Theorem 1.6.
f) True, by definition.

### Exercise 6

Consider a linear combination of the given matrices

$$\sum_{i=1,...,m} \sum_{j=1,...,n} a_{ij} E^{ij}$$

for some scalars $a_{ij} \in F$, and assume that it equals zero. Since, for fixed $i$ and $j$, the $(h, k)-$entry of $a_{ij}E^{ij}$ is:

$$(a_{ij}E^{ij})_{hk} = \begin{cases} a_{ij} & \text{if } h = i, \ j = k \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

it is clear that the linear combination above equals a matrix whose $(i, j)-$entry is $a_{ij}$:

$$\sum_{i=1,...,m} \sum_{j=1,...,n} a_{ij} E^{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}.$$

Since such matrix equals the zero matrix, necessarily $a_{ij} = 0$ for all $i = 1, ..., m$ and $j = 1, ..., n$. Therefore the given set of matrices is linearly independent.

## Exercise 15

If $u_1 = 0$ or $u_{k+1} \in \text{span}\{u_1, ..., u_k\}$ for some $k = 1, ..., n-1$, then 0 has a nontrivial representation as linear combination of vectors of $S$, i.e. $S$ is linearly dependent.

Conversely, assume $S$ is linearly dependent. If $u_i = 0$ for some index $i = 1, .., n$ (which after renumbering can be assumed to be 1), then there is nothing to prove. Therefore we can assume that $u_i \neq 0$ for all $i = 1, ..., n$. By hypothesis, there is a linear combination $a_1u_1 + ... + a_nu_n = 0$ for some $a_1, ..., a_n \in F$ not all zero. Let $k+1$ be the biggest index such that $a_{k+1} \neq 0$. Then we have $a_1u_1 + ... + a_{k+1}u_{k+1} = 0$ and we can rewrite it as

$$u_{k+1} = -\frac{1}{a_{k+1}}(a_1u_1 + ... + a_ku_k).$$

I.e., $u_{k+1} \in \text{span}\{u_1, ..., u_k\}$.

## Exercise 16

Assume that $S$ is linearly independent, and that $S_1$ is a (finite) subset of $S$. Then, by Corollary to Theorem 1.6, $S_1$ is linearly independent.

Assume that **every finite** subset of $S$ is linearly independent. Now consider any linear combination of vectors of $S$ that equals zero, i.e. a **finite** sum

$$\sum_{i=1,...,n} a_iv_i$$

for some $v_i \in S$ and $a_i \in F$. Consider the (finite) subset $S_1 := \{v_1, ..., v_n\}$ of $S$. By hypothesis, $S_1$ is linearly independent. Therefore, since

$$\sum_{i=1,...,n} a_iv_i = 0,$$

necessarily $a_i = 0$ for all $i = 1, ..., n$. This proves that $S$ is linearly independent.

## Exercise 18

Assume by contradiction that $S$ is linearly dependent. This means that we can find a nontrivial linear combination of polynomials in $S$ that is equal to zero:

$$\sum_{k=1}^{n} a_kp_k(x) = 0,$$

with $a_k \in F$ and $p_k(x)$ polynomial of degree $m_k$. We can assume that the coefficients of such linear combination are all nonzero. Because of the hypothesis on $S$, the degrees $m_1, ..., m_n$ are all distinct. Thus we can also assume, up to renumbering, that $m_1 < m_2 < ... < m_n$. Therefore $\sum_{k=1}^{n} a_kp_k(x)$ is a polynomial of degree $m_n$, and the leading coefficient is $a_nx^n$.

If the linear combiation is zero, by the identity principle for polynomials we must have in particular that $a_n = 0$, a contradiction. Therefore $S$ is linearly independent. $\square$