HW 13

Sec. 5.4

(4) W T-invariant; Then, by induction, assuming $T^{k-1}(W) \subset W$, we have $T^k(W) = T(T^{k-1}(W)) \subset$
$\subset T(W) \subset W$.
Since $g(T) = \sum_{i=0}^{m} a_i T^i$ for some $a_0, \dots, a_m \in F$, $T^i(W) \subset W$ $\forall i$, and W is closed under addition and scalar multiplication, we have that $(g(T))(W) \subset W$.

(6) Use thm. 5.22 (a):
(a) basis is $(z, T(z), T^2(z)) = \left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\-3\\3\\3\end{pmatrix}\right)$
since $T^3(z) = \begin{pmatrix}0\\-3\\3\\3\end{pmatrix} = 3T^2(z) - 3T(z)$.

(b) $(x^3, 6x)$
(c) $\left(\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\right)$.
(d) $= \left(\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\2 & 2\end{pmatrix}\right)$.

(11) (a) Say $\dim W = k$. Then, by thm. 5.22(a), $\{z, T(z), \dots, T^{k-1}(z)\}$ is a basis for W.
Take $w \in W$. Then $w = \sum_{i=0}^{k-1} a_i T^i(z)$ for some $a_i \in F$.

Thus $T(w) = \sum_{i=0}^{k-1} a_i T^{i+1}(z) = \underbrace{a_0 T(z) + \dots + a_{k-2} T^{k-1}(z)}_{\in W} +$
$+ a_{k-1} T^k(z)$. But also:
$T^k(z) \in W$ since W is T-cyclic.
Thus $T(w) \in W$.

(b) Let $Z \subset V$ be T-invariant, $v \in Z$.
Then $T(v) \in Z$, and by induction as in Ex. 4 we have $T^k(v) \in Z \ \forall k \geq 1$.
So $\{v, T(v), T^2(v), \dots\} \subset Z \Rightarrow$
$\Rightarrow W = \operatorname{span}\{v, T(v), T^2(v), \dots\} \subset Z$.

(15) By Cayley-Hamilton thm., $f(L_A) = T_0$.
By thm. 2.15, $f(L_A) = L_{f(A)}$.
Thus, $\forall v \in V$:
$$f(A) \cdot v = L_{f(A)}(v) = T_0(v) = 0.$$
Therefore $f(A) = O$.

(18) (a) $a_0 = f(0) = |A - 0 \cdot I| = |A|$ and we know:
A invertible $\Leftrightarrow |A| \neq 0$.

(b) By Cayley-Hamilton, $f(A) = O$. Thus:

$(-1)^n A^n + \dots + a_1 A + a_0 I = O \Rightarrow$ (multiply by A^{-1})

$\Rightarrow (-1)^n A^{n-1} + \dots + a_1 I + a_0 A^{-1} = O \Rightarrow$

$\Rightarrow A^{-1} = \left(-\frac{1}{a_0}\right) \cdot \left((-1) A^{n-1} + \dots + a_1 I\right).$

(c) $f(t) = -t^3 + 2t^2 + t - 2 \Rightarrow$

$\Rightarrow A^{-1} = \frac{1}{2} \cdot (-A^2 + 2A + I) = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1/2 & 3/2 \\ 0 & 0 & -1 \end{pmatrix}.$

Sec. 7.1

(2) (a) $f(t) = (2-t)^2 \qquad \lambda = 2 \qquad m = 2.$

$r_1 = 2 - rk(A - 2I) = 2 - 1 = 1$, so dot diagram is $\begin{array}{c}\bullet\\ \bullet\end{array}$.

and $J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

(b) $f(t) = (t-4)(t+1) \quad \lambda_1 = 4, \lambda_2 = -1$ both with multiplicity one $\Longrightarrow \quad J = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$ since A is diagonalizable.

(c) $f(t) = (t-2)^2(-t-1) \qquad \lambda_1 = 2 \quad m_1 = 2$

$\lambda_2 = -1 \quad m_2 = 1$

For λ_1, we have $r_1 = 2 - 1 = 1$.

For λ_1: $\begin{array}{c}\bullet\\ \bullet\end{array}$ $\qquad J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

For λ_2: $\bullet$

(d) $f(t) = (2-t)^2(3-t)^2 \qquad \lambda_1 = 2 \quad m_1 = 2$

$\lambda_2 = 3 \quad m_2 = 2.$

For λ_1, $r_1 = 2 - 1 = 1$

For λ_2, $r_1 = 2 - 0 = 2.$

For λ_1: $\begin{array}{c}\bullet\\ \bullet\end{array}$ $\qquad J = \left[\begin{array}{cc|cc} 2 & 1 & & \\ & 2 & & \\ \hline & & 3 & \\ & & & 3 \end{array}\right]$

For λ_2: $\bullet \; \bullet$

③ (a) $\beta = (1, x, x^2)$, $[T]_\beta = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix}$

$f(t) = (2-t)^3$. $\lambda = 2$, $m = 3$.

$r_1 = 3 - 2 = 1 \implies \begin{matrix} \bullet \\ \bullet \\ \bullet \end{matrix}$ $J = \begin{pmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{pmatrix}$

(b) $\beta = (1, t, t^2, e^t, te^t)$, $[T]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

$f(t) = -t^3(1-t)^2$

$\lambda_1 = 0$, $m_1 = 3$; $\lambda_2 = 1$, $m_2 = 1$

For λ_1: $r_1 = 1 \implies \begin{matrix} \bullet \\ \bullet \end{matrix}$

$J = \left[\begin{array}{cc|ccc} 1 & 1 & & & \\ & 1 & & & \\ \hline & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{array}\right]$

For λ_2: $r_1 = 1 \implies \begin{matrix} \bullet \\ \bullet \\ \bullet \end{matrix}$

(c) β standard, $[T]_\beta = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

$f(t) = (t-1)^4$ $\lambda = 1$, $m = 4$.

$r_1 = 4 - 2 = 2$,
$r_2 = 2 - 0 = 2$ $\implies \begin{matrix} \bullet & \bullet \\ \bullet & \bullet \end{matrix}$ $J = \left(\begin{array}{cc|cc} 1 & 1 & & \\ & 1 & & \\ \hline & & 1 & 1 \\ & & & 1 \end{array}\right)$

(d) β standard, $f(t) = (t-1)(t-3)^3$

$\lambda_1 = 1$, $m_1 = 1$, $\lambda_2 = 3$, $m_2 = 3$.

For λ_2: $r_1 = 3 \implies \bullet\ \bullet\ \bullet$ $J = \begin{bmatrix} 3 & & & \\ & 3 & & \\ & & 3 & \\ & & & 1 \end{bmatrix}$

For λ_1: $\bullet$

④ Let $W = \mathrm{span}(\gamma)$. Then W is $(T-\lambda I)$-invariant.
Take any $w \in W$. Then:

$$T(w) = T(w) - \lambda \cdot w + \lambda \cdot w = \underbrace{(T-\lambda I)(w)}_{\in W} + \underbrace{\lambda w}_{\substack{\in W \\ \text{since } W \\ \text{subspace}}} \in W$$

Thus $T(W) \subset W$.

⑦ (a) We need to show that $N(U^k) \subset N(U^{k+1})$ $\forall k$.
Say $v \in N(U^k)$. Then $U^k(v) = 0 \Rightarrow$

$$\Rightarrow \underbrace{U(U^k(v))}_{= U^{k+1}(v)} = U(0) = 0 \Rightarrow v \in N(U^{k+1}).$$

(b) Say, by induction, that $rk(U^m) = rk(U^k)$.
~~Since~~ $U^k(V) \subset U^m(V)$, we have $U^m(V) = U^k(V)$.
Then $U^m(V) = U^k(V) \xRightarrow[\text{applying } U \text{ both sides}]{} U^{m+1}(V) = U^{k+1}(V)$

Since by hypothesis $U^{m+1}(V) \overset{!}{=} U^m(V)$, we have
$U^{k+1}(V) = U^m(V)$.

(c) By (b), $rk(U^m) = rk(U^k)$ $\forall k \geq m$.
By rank-nullity thm. $\mathrm{null}(U^m) = \mathrm{null}(U^k)$ $\forall k \geq m$
Since by (a): $N(U^m) \subset N(U^k)$, thus
also implies $N(U^m) = N(U^k)$.

(d) $K_\lambda = \bigcup_k N((T-\lambda I)^k)$. By (a), $N(T-\lambda I) \subset$
$\subset N((T-\lambda I)^2) \subset \ldots \subset N((T-\lambda I)^m)$. By (c),
$N((T-\lambda I)^m) = N((T-\lambda I)^k)$ $\forall k \geq m+1$. Thus
$K_\lambda = N((T-\lambda I)^m)$.