HW 12

4.5.7 Yes

4.5.12 (i) $A = E_1 \Rightarrow \delta(E_1) = -1 \Rightarrow \delta(AB) = -\delta(B) = \delta(A)\delta(B)$

(ii) $A = E_2 \Rightarrow \delta(E_2) = k \Rightarrow \delta(AB) = k\,\delta(B) = \delta(A)\delta(B)$.

(iii) $A = E_3 \Rightarrow \delta(E_3) = 1 \Rightarrow \delta(AB) = \delta(B) = \delta(A)\delta(B)$.

4.5.13. $\forall A \in M_{2\times 2}(F)$, $A = (a_1, a_2)$, $v = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. a_i : columns of A

(i) $\det(a_1, a_2 + kv) = \det(a_1, a_2) + k\det(a_1, v)$

~~(ii) $\det(a_1 + kv, a_2) = \det(a_1, a_2) + k\det(v, a_2)$.~~

5.1.2

(a) $T\begin{pmatrix}1\\2\end{pmatrix} = -\begin{pmatrix}2\\3\end{pmatrix}$, $T\begin{pmatrix}2\\3\end{pmatrix} = 2\begin{pmatrix}1\\2\end{pmatrix}$. $\therefore [T]_\beta = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$

(c) $T\begin{pmatrix}0\\1\\1\end{pmatrix} = -\begin{pmatrix}0\\1\\1\end{pmatrix}$; $T\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$; $T\begin{pmatrix}1\\0\\2\end{pmatrix} = -\begin{pmatrix}1\\0\\2\end{pmatrix}$ $\therefore [T]_\beta = \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$

(f) $T\begin{pmatrix}1 & 0\\1 & 0\end{pmatrix} = -3\begin{pmatrix}1 & 0\\1 & 0\end{pmatrix}$

$T\begin{pmatrix}-1 & 2\\0 & 0\end{pmatrix} = +\begin{pmatrix}-1 & 2\\0 & 0\end{pmatrix}$ $\qquad \therefore [T]_\beta = \begin{bmatrix} -3 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$.

$T\begin{pmatrix}1 & 0\\2 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\2 & 0\end{pmatrix}$

$T\begin{pmatrix}-1 & 0\\0 & 2\end{pmatrix} = \begin{pmatrix}-1 & 0\\0 & 2\end{pmatrix}$

5.1.3. (b). (i) $\lambda = 1, 2, 3$

(ii) $\left\{ t\begin{pmatrix}1\\1\\-1\end{pmatrix} : t \in \mathbb{R} \right\}$, $\left\{ t\begin{pmatrix}1\\-1\\0\end{pmatrix} : t \in \mathbb{R} \right\}$, $\left\{ t\begin{pmatrix}1\\0\\-1\end{pmatrix} : t \in \mathbb{R} \right\}$.

(iii) $\beta = \left\{ \begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix} \begin{pmatrix}1\\0\\-1\end{pmatrix} \right\}$.

(iv) $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$. $D = Q^{-1}AQ = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$.

(d) (i) $\lambda = 0, 1$

(ii). $\left\{ t\begin{pmatrix}1\\4\\2\end{pmatrix} : t \in \mathbb{R} \right\}$; $\left\{ t\begin{pmatrix}1\\0\\1\end{pmatrix} + s\begin{pmatrix}0\\1\\0\end{pmatrix} : s, t \in \mathbb{R} \right\}$.

(iii) $\beta = \left\{ \begin{pmatrix}1\\4\\2\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\}$.

(iv). $Q = \begin{pmatrix} 1 & 1 & 0 \\ 4 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$, $D = Q^{-1}AQ = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}$.

5.1.7. (a) $[T]_\beta = [I]_{\gamma}^{\beta} [T]_\gamma [I]_\beta^\gamma$ so ~~$[I]_\beta$~~ if we put $P = [I]_\gamma^\beta$ then $[T]_\beta = P[T]_\gamma P^{-1}$. so $\det([T]_\beta) = \det([T]_\gamma)$.

(b) T is invertible iff $[T]_\beta$ is invertible

iff $\det([T]_\beta) \neq 0$ (a theorem).

iff $\det(T) \neq 0$ (definition)

(c) Note that $[T^{-1}]_\beta = ([T]_\beta)^{-1}$. Since $\det(A^{-1}) = (\det(A))^{-1}$ for a square matrix A.

we know that $\det(T^{-1}) = \det([T^{-1}]_\beta) = \det(([T]_\beta)^{-1}) = (\det([T]_\beta))^{-1} = (\det(T))^{-1}$

(d) Since $[TU]_\beta = [T]_\beta [U]_\beta$, we know that

$\det(TU) = \det([TU]_\beta) = \det([T]_\beta [U]_\beta) = \det([T]_\beta) \det([U]_\beta)$

$= \det(T)\det(U)$.

(e) $[T - \lambda I_V]_\beta = [T]_\beta - [\lambda I_V]_\beta = [T]_\beta - \lambda I$

So, $\det(T - \lambda I_V) = \det([T - \lambda I_V]_\beta) = \det([T]_\beta - \lambda I)$.

~~5.1.8. (a) T is invertible $\Leftrightarrow$ $\det(T) \neq 0$~~

~~$\Leftrightarrow$~~

~~$\lambda = 0$ is not a root of $f(x) = \det(T - \lambda I_V)$~~

~~$\Leftrightarrow$ $\lambda = 0$ is not an eigenvalue of T.~~

5.1.8. (a) ~~T is invertible $\Leftrightarrow$ det(T) ≠ 0~~.

"$\Rightarrow$" Suppose zero is an eigenvalue of T, then there exists $x \in V$ s.t. $x \neq 0$ $Tx = 0$. then $N(T) \neq \{0\}$. So T is not invertible

"$\Leftarrow$" Suppose T is not invertible. then $\exists\, x \neq 0$. $Tx = 0$.

(b). $Tx = \lambda x \quad \lambda \neq 0 \Rightarrow T^{-1}Tx = \lambda T^{-1}x \Rightarrow x = \lambda T^{-1}x$

$\Rightarrow T^{-1}x = \lambda^{-1}x$.

~~Let~~

5.2.8 Let $v_1, \cdots, v_{n-1}$ span E_{λ_1}, then $v_1, \cdots, v_{n-1}$ is linearly independent.

Let $v_n \in F^n$ s.t. $A v_n = \lambda_2 v_n \quad (v_n \neq 0)$

Claim, $v_1, \cdots, v_{n-1}, v_n$ linearly independent : (*)

Let $k_1 v_1 + \cdots + k_{n-1} v_{n-1} + k_n v_n = 0$. ①

$\Rightarrow k_1 \lambda_1 v_1 + \cdots + k_{n-1}\lambda_1 v_{n-1} + k_n \lambda_2 v_n = 0$ ②

$\lambda_1 \cdot ① - ② \Rightarrow k_n(\lambda_1 - \lambda_2) v_n = 0. \Rightarrow k_n = 0$.

So $k_1 v_1 + \cdots + k_{n-1} v_{n-1} = 0$, because $v_1 \cdots v_{n-1}$ linearly independent, $k_1 = \cdots = k_{n-1} = 0$. So (*) is true.

Now put $P = (v_1, \cdots, v_{n-1}, v_n)$. then $AP = (\lambda_1 v_1, \cdots \lambda_1 v_{n-1}, \lambda_2 v_n)$

$= P \begin{pmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_1 & \\ & & & \lambda_2 \end{pmatrix}$.

5.2.10. $[T]_\beta = \begin{bmatrix} T_{11} & & * \\ & \ddots & \\ 0 & & T_{nn} \end{bmatrix}$.

$\Rightarrow \det(T - \lambda I_V) = \det([T]_\beta - \lambda I) = (T_{11} - \lambda) \cdots (T_{nn} - \lambda)$.

5.2.12. (a) the eigenspace of T corresponding to λ

$$= N(T-\lambda I)$$

$$= \{x \in V : Tx = \lambda x\}$$

$$= \{x \in V : T^{-1}x = \lambda^{-1}x\} = N(T^{-1} - \lambda^{-1}I)$$

(b) T is diagonalizable, $\Leftrightarrow$ $\exists$ a ordered basis $\beta = \{\beta_1, \cdots, \beta_n\}$ s.t. $\cancel{T\beta_i = \lambda_i}$ $[T]_\beta = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$\Leftrightarrow$ $\exists$ a ordered basis β s.t. $[T^{-1}]_\beta = [T]_\beta^{-1} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}^{-1} = \begin{pmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_n^{-1} \end{pmatrix}$