Now let E_1, E_2 be products of elementary matrices such that $E_1 C$ and $E_2 A$ are in rref form. In particular, they are upper triangular.

Now consider:

$$\begin{bmatrix} E_2 & 0 \\ 0 & I \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & E_1 \end{bmatrix} \cdot \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \begin{bmatrix} E_2 A & E_2 B \\ 0 & E_1 C \end{bmatrix}.$$

Then since $E_2 A$ and $E_1 C$ are upper triangular, $\begin{bmatrix} E_2 A & E_2 B \\ 0 & E_1 C \end{bmatrix}$ is also upper triangular, and it has determinant equal to:

$$\det(E_2 A) \cdot \det(E_1 C) \stackrel{!}{=} \det E_2 \cdot \det A \cdot \det E_1 \cdot \det C.$$

On the other hand,

$$\det\left[\begin{pmatrix} E_2 & 0 \\ 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & 0 \\ 0 & E_1 \end{pmatrix} \cdot \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\right] = (\text{by Ex. } 5)$$

$$= \det E_2 \cdot \det E_1 \cdot \det M.$$

Since $\det E_1, \det E_2 \neq 0$, and:

$\det E_2 \det E_1 \det M = \det E_2 \det A \det E_1 \cdot \det C$, we can simplify and obtain:

$$\det M = \det A \cdot \det C$$

(5) By induction on $m = rk(I)$.

$m=1$: $M = \left[\begin{array}{c|c} A & b \\ \hline 0 \cdots 0 & 1 \end{array}\right]$

By cofactor expansion along the last row,

$$\det(M) = (-1)^{n+n} \cdot \det(\tilde{M}_{nn}) = (-1)^{2n} \det(A) = \det(A).$$

Assume it's true for $m = r-1$.

$m=2$: $M = \left[\begin{array}{c|c} A & B \\ \hline 0 & I_r \end{array}\right]$. By cofactor expansion along last row,

$$\det(M) = (-1)^{n+n} \cdot \det(\tilde{M}_{nn}) =$$

$$= \det \left[\begin{array}{c|c} A & \tilde{B} \\ \hline 0 & I_{r-1} \end{array}\right]$$ where $\tilde{B}$ is obtained from B by removing the last column.

$= \det(A)$.

by induction hypothesis

(6) As a special case of Ex. 5, we have that:

$$\det \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix} = \det A.$$

Similarly, $\det \begin{bmatrix} I & 0 \\ 0 & A \end{bmatrix} = \det A.$

But $\det(M^{r-1}) = 0$ implies $\det(M) = 0$ by induction.
Therefore $\det(M) = 0$ in either case.

⑬ (a) by induction on n.
$n = 2$:
$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{C}).$$
$$\det(\overline{M}) = \det\begin{bmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{bmatrix} = \overline{a}\,\overline{d} - \overline{b}\,\overline{c} = \overline{ad - bc} =$$
$$= \overline{\det(M)}.$$
Assume it's true for $n = r-1$.
$n = r$:
$$\det \overline{M} = \sum_{j=1}^{r} (-1)^{1+j}\, \overline{M}_{1j} \cdot \det\left(\widetilde{\overline{M}}_{1j}\right) \underset{\text{by induction}}{=}$$
since $\widetilde{M}_{1j} \in M_{(r-1)\times(r-1)}$
$$= \sum_{j=1}^{r} (-1)^{1+j}\, \overline{M_{1j}} \cdot \overline{\det(\widetilde{M}_{1j})} =$$
$$= \overline{\sum_{j=1}^{r} (-1)^{1+j} M_{1j} \cdot \det(\widetilde{M}_{1j})} = \overline{\det(M)}.$$

(b) If Q unitary, then $QQ^* = I \Rightarrow$
$$\Rightarrow 1 = \det(I) = \det(QQ^*) = \det(Q) \cdot \det(\overline{Q^t}) =$$
$$= \det(Q) \cdot \overline{\det(Q^t)} = \det(Q) \cdot \overline{\det(Q)} =$$
$$= |\det(Q)|.$$

(15) A, B similar $\Rightarrow \exists\, Q \in M_{n\times n}(F)$ invertible s.t. $B = Q^{-1}AQ$.

Then $\det(B) = \det(Q^{-1}AQ) = \det(Q^{-1}) \cdot \det A \det Q$

$= \underbrace{(\det Q)^{-1} \cdot \det Q}_{=1} \cdot \det A = \det A$.

(18) Say A is elementary of type 2, obtained by multiplying a row of I by $k \in F$.
Then $\det(A) = k$, and AB is obtained by multiplying a row of B by k.
Thus: $\det(AB) = k \cdot \det(B) = \det(A) \cdot \det(B)$.
If A is of type 3, $\det A = 1$, and AB is obtained by adding to a row of B a multiple of another row.
Thus: $\det(AB) = \det(B) = 1 \cdot \det(B) = \det A \cdot \det B$

(4) (b) $\det \begin{bmatrix} -1 & 3 & 2 \\ 0 & 4 & 9 \\ 0 & 8 & 9 \end{bmatrix} = (-1) \cdot \begin{vmatrix} 4 & 9 \\ 8 & 9 \end{vmatrix} =$

$= (-1) \cdot (36 - 72) = 36$.

(g) $\det \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 4 & -1 & 1 \\ 0 & 3 & 4 & -5 \end{bmatrix} = \det \begin{bmatrix} 1 & -5 & 11 \\ 4 & -1 & 1 \\ 3 & 4 & -5 \end{bmatrix} =$

$= \det \begin{bmatrix} 1 & -5 & 11 \\ 0 & 19 & -43 \\ 0 & 0 & 5 \end{bmatrix} = \det \begin{bmatrix} 19 & -43 \\ 0 & 5 \end{bmatrix} = 95.$

HW 12

4.5.7 Yes

4.5.12 (i) $A = E_1 \Rightarrow \delta(E_1) = -1 \Rightarrow \delta(AB) = -\delta(B) = \delta(A)\,\delta(B)$

(ii) $A = E_2 \Rightarrow \delta(E_2) = k \Rightarrow \delta(AB) = k\,\delta(B) = \delta(A)\,\delta(B)$.

(iii) $A = E_3 \Rightarrow \delta(E_3) = 1 \Rightarrow \delta(AB) = \delta(B) = \delta(A)\,\delta(B)$.

4.5.13. $\forall A \in M_{2\times 2}(F)$, $A = (a_1, a_2)$, $v = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. a_i : columns of A

(i) $\det(a_1, a_2 + kv) = \det(a_1, a_2) + k\det(a_1, v)$

(ii) $\det(a_1 + kv, a_2) = \det(a_1, a_2) + k\det(v, a_2)$.

5.1.2

(a) $T\begin{pmatrix} 1 \\ 2 \end{pmatrix} = -\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $T\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. $\therefore [T]_\beta = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$

(c) $T\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; $T\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$; $T\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ $\therefore [T]_\beta = \begin{bmatrix} -1 & & \\ & 1 & \\ & & -1 \end{bmatrix}$

(f) $T\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = -3\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

$T\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} = +\begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}$ $\therefore [T]_\beta = \begin{bmatrix} -3 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$.

$T\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$

$T\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

5.1.3. (b). (i) $\lambda = 1, 2, 3$

(ii) $\left\{ t\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} : t \in \mathbb{R} \right\}$, $\left\{ t\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$, $\left\{ t\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} : t \in \mathbb{R} \right\}$.

(iii) $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$.

(iv) $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}$. $D = Q^{-1}AQ = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}$.

(d) (i) $\lambda = 0, 1$

(ii). $\left\{ t\begin{pmatrix}1\\4\\2\end{pmatrix} : t \in \mathbb{R} \right\}$; $\left\{ t\begin{pmatrix}1\\0\\1\end{pmatrix} + s\begin{pmatrix}0\\1\\0\end{pmatrix} : s, t \in \mathbb{R} \right\}$.

(iii) $\beta = \left\{ \begin{pmatrix}1\\4\\2\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\}$.

(iv). $Q = \begin{pmatrix} 1 & 1 & 0 \\ 4 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$, $D = Q^{-1}AQ = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix}$.

5.1.7. (a) $[T]_\beta = [I]_\gamma^\beta [T]_\gamma [I]_\beta^\gamma$ so ~~$[I]_\beta$~~ if we put $P = [I]_\gamma^\beta$ then $[T]_\beta = P[T]_\gamma P^{-1}$. so $\det([T]_\beta) = \det([T]_\gamma)$.

(b) T is invertible iff $[T]_\beta$ is invertible

iff $\det([T]_\beta) \neq 0$ (a theorem).

iff $\det(T) \neq 0$ (definition)

(c) Note that $[T^{-1}]_\beta = ([T]_\beta)^{-1}$. Since $\det(A^{-1}) = (\det(A))^{-1}$ for a square matrix A.
we know that $\det(T^{-1}) = \det([T^{-1}]_\beta) = \det(([T]_\beta)^{-1}) = (\det([T]_\beta))^{-1} = (\det(T))^{-1}$

(d) Since $[TU]_\beta = [T]_\beta [U]_\beta$, we know that

$$\det(TU) = \det([TU]_\beta) = \det([T]_\beta [U]_\beta) = \det([T]_\beta)\det([U]_\beta)$$
$$= \det(T)\det(U).$$

(e) $$[T - \lambda I_V]_\beta = [T]_\beta - [\lambda I_V]_\beta = [T]_\beta - \lambda I$$

So, $\det(T - \lambda I_V) = \det([T - \lambda I_V]_\beta) = \det([T]_\beta - \lambda I)$.

~~5.1.8. (a) T is invertible $\Leftrightarrow$ $\det(T) \neq 0$~~

~~$\Leftrightarrow$ $\lambda = 0$ is not a root of $f(\lambda) = \det(T - \lambda I_V)$~~

~~$\Leftrightarrow$ $\lambda = 0$ is not an eigenvalue of T.~~