HW 11

4.3

⑦ $Ax = b$, $A = \begin{bmatrix} 3 & 1 & 1 \\ -2 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 4 \\ 12 \\ -8 \end{bmatrix}$

$\det A = \begin{vmatrix} -2 & -1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ -2 & -1 \end{vmatrix} = (-4+1) + (-3+2) = -4 \neq 0$

$x_1 = \dfrac{\begin{vmatrix} 4 & 1 & 1 \\ 12 & -1 & 0 \\ -8 & 2 & 1 \end{vmatrix}}{-4} = -\frac{1}{4}\left[\begin{vmatrix} 12 & -1 \\ -8 & 2 \end{vmatrix} + \begin{vmatrix} 4 & 1 \\ 12 & -1 \end{vmatrix} \right] =$

$= -\frac{1}{4} \cdot \left((24-8) + (-4-12) \right) = 0$

$x_2 = \dfrac{\begin{vmatrix} 3 & 4 & 1 \\ -2 & 12 & 0 \\ 1 & -8 & 1 \end{vmatrix}}{-4} = \dots = -12$, $x_3 = \dfrac{\begin{vmatrix} 3 & 1 & 4 \\ -2 & -1 & 12 \\ 1 & 2 & -8 \end{vmatrix}}{-4} = \dots = 16$

⑧ If $A = [a_1 \ \cdots \ a_n]$ has columns $a_1, \dots, a_n$,

then $A^t = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ has rows $a_1, \dots, a_n$.

Thus: $\det [a_1 \cdots a_{r-1} \ u+kv \ a_{r+1} \cdots a_n] =$

$= \det \left([a_1 \cdots a_{r-1} \ u+kv \ a_{r+1} \cdots a_n]^t \right) =$

$= \det \left(\begin{bmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u+kv \\ a_{r+1} \\ \vdots \\ a_n \end{bmatrix} \right) = \det \begin{bmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{bmatrix} + k \cdot \det \begin{bmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{bmatrix} =$

$$= \det\left(\begin{bmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{bmatrix}^t\right) + K\det\left(\begin{bmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{bmatrix}^t\right) =$$

$$= \det[a_1 \cdots a_{r-1}\ u\ a_{r+1} \cdots a_n] + K\cdot\det[a_1 \cdots a_{r-1}\ v\ a_{r+1} \cdots a_n]$$

for any $K \in F$, $u, v, a_i \in F^n$.

(9) Say $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & & \vdots \\ 0 & & \ddots & \vdots \\ & & & a_{nn} \end{bmatrix}$ is upper triangular.

By 4.2, Exerc. 23, $\det(A) = \prod_{i=1}^{n} a_{ii}$.
By Thm. 4.7,
A is invertible $\iff \det(A) \neq 0$,
and $\det(A) = \prod_{i=1}^{n} a_{ii} \neq 0 \iff a_{ii} \neq 0$
$\forall i = 1, \ldots, n$.

(10) By induction:
$K=1$: $M = O \Rightarrow \det(M) = \det(O) = 0$.
Assume the result true for $K = r-1$.
$K = r$: $M^r = O \Rightarrow \det(M^r) = \det(O) = 0$.
By Thm. 4.7, $0 = \det(M^r) =$

$= \det(M \cdot M^{r-1}) = \det(M)\cdot\det(M^{r-1})$.
Thus:
either $\det(M) = 0$, or $\det(M^{r-1}) = 0$.

But $\det(M^{r-1}) = 0$ implies $\det(M) = 0$ by induction.
Therefore $\det(M) = 0$ in either case.

(13) (a) by induction on n.
$n = 2$:
$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{C}).$$

$$\det(\overline{M}) = \det \begin{bmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{bmatrix} = \overline{a}\,\overline{d} - \overline{b}\,\overline{c} = \overline{ad - bc} =$$

$$= \overline{\det(M)}.$$

Assume it's true for $n = r-1$.

$n = r$:
$$\det \overline{M} = \sum_{j=1}^{r} (-1)^{1+j}\, \overline{M_{1j}} \cdot \det\left(\widetilde{\overline{M}}_{1j}\right) =$$

by induction since $\widetilde{M}_{1j} \in M_{(r-1)\times(r-1)}$

$$= \sum_{j=1}^{r} (-1)^{1+j}\, \overline{M_{1j}} \cdot \overline{\det(\widetilde{M}_{1j})} =$$

$$= \overline{\sum_{j=1}^{r} (-1)^{1+j} M_{1j} \cdot \det(\widetilde{M}_{1j})} = \overline{\det(M)}.$$

(b) If Q unitary, then $QQ^* = I \Rightarrow$

$$\Rightarrow 1 = \det(I) = \det(QQ^*) = \det(Q) \cdot \det(\overline{Q^t}) =$$
$$= \det(Q) \cdot \overline{\det(Q^t)} = \det(Q) \cdot \overline{\det(Q)} =$$
$$= |\det(Q)|.$$

(15) A, B similar $\Rightarrow \exists\, Q \in M_{n\times n}(F)$ invertible s.t. $B = Q^{-1}AQ$.

Then $\det(B) = \det(Q^{-1}AQ) = \det(Q^{-1})\cdot \det A \det Q$

$= \underbrace{(\det Q)^{-1} \cdot \det Q}_{=1} \cdot \det A = \det A.$

(18) Say A is elementary of type 2, obtained by multiplying a row of I by $k \in F$.

Then $\det(A) = k$, and AB is obtained by multiplying a row of B by k.

Thus: $\det(AB) = k \cdot \det(B) = \det(A)\cdot\det(B)$.

If A is of type 3, $\det A = 1$, and AB is obtained by adding to a row of B a multiple of another row.

Thus: $\det(AB) = \det(B) = 1 \cdot \det(B) = \det A \cdot \det B$

(4) (b) $\det \begin{bmatrix} -1 & 3 & 2 \\ 0 & 4 & 9 \\ 0 & 8 & 9 \end{bmatrix} = (-1)\cdot \begin{vmatrix} 4 & 9 \\ 8 & 9 \end{vmatrix} =$

$= (-1)\cdot(36 - 72) = 36.$

(g) $\det \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 4 & -1 & 1 \\ 0 & 3 & 4 & -5 \end{bmatrix} = \det \begin{bmatrix} 1 & -5 & 11 \\ 4 & -1 & 1 \\ 3 & 4 & -5 \end{bmatrix} =$

$= \det \begin{bmatrix} 1 & -5 & 11 \\ 0 & 19 & -43 \\ 0 & 0 & 5 \end{bmatrix} = \det \begin{bmatrix} 19 & -43 \\ 0 & 5 \end{bmatrix} = 95.$