MAT 310
HW 1

Section 1.2

Exercise 1

a) True, by definition.
b) False, by Corollary 1.
c) False. If $x = 0$, then $ax = bx$ for all $a, b \in F$.
d) False. If $a = 0$, then $ax = ay$ for all vectors x and y.
e) True. Vectors in F^n can be written as column vectors $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ which can be regarded as matrices with n rows and 1 column.
f) False.
g) False.
h) False. For example, $f = x^2 + x$ and $g = -x^2$ are polynomials of degree 2, but $f + g = x$ is a polynomial of degree 1.
i) True. If $f = a_n x^n +$ lower order terms, $a_n \neq 0$, and c is a nonzero scalar, then $cf = ca_n x^n +$ lower order terms, and $ca_n \neq 0$. Therefore cf is a polynomial of degree n.
j) True.
k) True, by definition.

Exercise 4

a) $\begin{bmatrix} 7 & 3 & 2 \\ -4 & 3 & 9 \end{bmatrix}$
b) $\begin{bmatrix} 1 & -1 \\ 3 & -5 \\ 3 & 8 \end{bmatrix}$
c) $\begin{bmatrix} 8 & 20 & -12 \\ 4 & 0 & 28 \end{bmatrix}$
d) $\begin{bmatrix} 30 & -20 \\ -15 & 10 \\ -5 & -40 \end{bmatrix}$

e) $2x^4 + x^3 + 2x^2 - 2x + 10$
f) $-x^3 + 7x^2 + 4$
g) $10x^7 - 30x^4 + 40x^2 - 15x$
h) $3x^5 - 6x^3 + 12x + 6$

Exercise 9

Proof of Corollary 1: Assume that there are two vectors 0_A and 0_B, with the property that $x + 0_A = x$ and $x + 0_B = x$ for all $x \in V$. Fix any $x \in V$. Then we have $x + 0_A = x + 0_B$. Therefore, by the Cancellation Law for vectors, $0_A = 0_B$.

Proof of Corollary 2: Assume that, for some $x \in V$ fixed, there are two vectors y_1 and y_2 with the property that $x + y_1 = 0$ and $x + y_2 = 0$. Then $x + y_1 = x + y_2$ and, by the Cancellation Law for vectors, $y_1 = y_2$.

Proof of Theorem 1.2c): Fix any $a \in F$. Then $a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$ (by VS 3 and VS 7). Thus, by the Cancellation Law for vectors, $0 = a \cdot 0$.

Exercise 18

V is NOT a vector space. To show this, it is enough to check that one of the axioms VS 1,...,VS 8 fails. For example, + is not commutative. In fact:

$$(1,2) + (3,0) = (7,2) \neq (5,6) = (3,0) + (1,2).$$

Exercise 21

We need to check the eight properties of addition and scalar multiplication. In what follows, $(v_1, w_1), (v_2, w_2), (v_3, w_3)$ are any elements of Z, and a, b are any elements of the field F.

VS 1: $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$ which equals (by commutativity of addition in V and W) $(v_2 + v_1, w_2 + w_1) = (v_2, w_2) + (v_1, w_1)$.

VS 2: $((v_1, w_1) + (v_2, w_2)) + (v_3, w_3) = (v_1 + v_2, w_1 + w_2) + (v_3, w_3) = ((v_1 + v_2) + v_3, (w_1 + w_2) + w_3)$ which equals (by associativity of addition in V and W) $(v_1 + (v_2 + v_3), w_1 + (w_2 + w_3)) = (v_1, w_1) + ((v_2, w_2) + (v_3, w_3))$.

VS 3: Let us denote by 0_V and 0_W the zero vectos of V and W respectively. Then $(0_V, 0_W)$ is the zero vector for Z. Namely: $(0_V, 0_W) + (v_1, w_1) = (0_V + v_1, 0_W + w_1) = (v_1, w_1)$.

VS 4: Since V and W are vector spaces, given $v_1 \in V, w_1 \in W$, there exist additive inverses $-v_1 \in V, -w_1 \in W$. Then $(-v_1, -w_1)$ is the additive inverse of (v_1, w_1). Namely: $(v_1, w_1) + (-v_1, -w_1) = (v_1 - v_1, w_1 - w_1) = (0, 0)$.

VS 5: $1 \cdot (v_1, w_1) = (1 \cdot v_1, 1 \cdot w_1) = (v_1, w_1)$ since 1 is the unit scalar for V and W.

VS 6: $(ab)(v_1, w_1) = ((ab)v_1, (ab)w_1) = (a(bv_1), a(bw_1)) = a(bv_1, bw_1) = a(b(v_1, w_1))$.

VS 7: $a((v_1, w_1)+(v_2, w_2)) = a(v_1+v_2, w_1+w_2) = (a(v_1+v_2), a(w_1+w_2)) = (av_1+av_2, aw_1+aw_2) = a(v_1, w_1) + a(v_2, w_2)$.

VS 8: $(a+b)(v_1, w_1) = ((a+b)v_1, (a+b)w_1) = (av_1 + bv_1, aw_1 + bw_1) = a(v_1, w_1) + b(v_1, w_1)$.

Section 1.3

Exercise 6

First note that the element in the i−th row and j−th column of $aA + bB$ is $(aA + bB)_{ij} = aA_{ij} + bB_{ij}$, just by definition of sum of matrices and scalar product. Therefore, we have that

$$\begin{aligned} \text{tr}(aA + bB) &= (aA + bB)_{11} + (aA + bB)_{22} + \ldots + (aA + bB)_{nn} = \\ &= (aA_{11} + bB_{11}) + (aA_{22} + bB_{22}) + \ldots + (aA_{nn} + bB_{nn}) = \\ &= a \cdot (A_{11} + A_{22} + \ldots + A_{nn}) + b \cdot (B_{11} + B_{22} + \ldots + B_{nn}) = a \cdot \text{tr}(A) + b \cdot \text{tr}(B). \end{aligned}$$

Exercise 9

$$\begin{aligned} W_1 \cap W_3 &= \{(a_1, a_2, a_3) \in R^3 \ : \ a_1 = 3a_2, \ a_3 = -a_2, \ 2a_1 - 7a_2 + a_3 = 0\} = \\ &= \{(a_1, a_2, a_3) \in R^3 \ : \ a_1 = 3a_2, \ a_3 = -a_2, \ 2(3a_2) - 7a_2 + (-a_2) = 0\} = \\ &= \{(a_1, a_2, a_3) \in R^3 \ : \ a_1 = 3a_2, \ a_3 = -a_2, \ -2a_2 = 0\} = \{(0,0,0)\} \end{aligned}$$

is the zero vector space, so it is clearly a vector space.

$$\begin{aligned} W_1 \cap W_4 &= \{(a_1, a_2, a_3) \in R^3 \ : \ a_1 = 3a_2, \ a_3 = -a_2, \ a_1 - 4a_2 - a_3 = 0\} = \\ &= \{(a_1, a_2, a_3) \in R^3 \ : \ a_1 = 3a_2, \ a_3 = -a_2, \ 3a_2 - 4a_2 - (-a_2) = 0\} = \\ &= \{(a_1, a_2, a_3) \in R^3 \ : \ a_1 = 3a_2, \ a_3 = -a_2\} = W_1 \end{aligned}$$

is W_1 itself (this means that $W_1 \subset W_4$). W_1 is a vector space. Namely:
1) $(0,0,0) \in W_1$ since it satisfies the equations $a_1 = 3a_2, \ a_3 = -a_2$.
2) Note that $W_1 = \{(3t, t, -t) \in R^3 \ : \ t \in R\}$. Thus, given two vectors $(3t, t, -t), (3s, s, -s) \in W_1$ for some $t, s \in R$, their sum is $(3(t+s), t+s, -(t+s))$ with $t+s \in R$, and therefore belongs to W_1.
3) Given a vector $(3t, t, -t) \in W_1$ for some $t \in R$ and a scalar $c \in R$, $c(3t, t, -t) = (3(tc), tc, -(tc))$ with $tc \in R$, and therefore belongs to W_1.

$$W_3 \cap W_4 = \{(a_1, a_2, a_3) \in R^3 \ : \ 2a_1 - 7a_2 + a_3 = 0, \ a_3 = a_1 - 4a_2\} =$$

$$= \{(a_1, a_2, a_3) \in R^3 \ : \ 2a_1 - 7a_2 + a_1 - 4a_2 = 0\} =$$
$$= \{(a_1, a_2, a_3) \in R^3 \ : \ 3a_1 - 11a_2 = 0\}.$$

$W_3 \cap W_4$ is a vector space. Namely:
1) The zero vector belongs to $W_3 \cap W_4$ since it satisfies the equation $3a_1 - 11a_2 = 0$.
2) If $(a_1, a_2, a_3), (b_1, b_2, b_3) \in W_3 \cap W_4$, then $3a_1 - 11a_2 = 0$ and $3b_1 - 11b_2 = 0$. Therefore the sum vector belongs to $W_3 \cap W_4$, since the equation $3(a_1 + b_1) - 11(a_2 + b_2) = 0$ is satisfied.
3) If $(a_1, a_2, a_3) \in W_3 \cap W_4$, then $3a_1 - 11a_2 = 0$. Say that $c \in R$ is a scalar. Then $c(a_1, a_2, a_3) \in W_3 \cap W_4$ since the equation $3(ca_1) - 11(ca_2) = 0$ is satisfied.

Exercise 10

To prove both that W_1 is a subspace and W_2 is not, we can use Theorem 1.3. Consider W_1 first.
The zero vector of F^n is $(0, ..., 0)$. Since $0 + ... + 0 = 0$, the zero vector belongs to W_1.
Now say that $(a_1, ..., a_n), (b_1, ..., b_n) \in W_1$. Then $(a_1, ..., a_n) + (b_1, ..., b_n) = (a_1 + b_1, ..., a_n + b_n)$. Since $(a_1 + b_1) + ... + (a_n + b_n) = (a_1 + ... + a_n) + (b_1 + ...b_n) = 0 + 0 = 0$, the sum vector $(a_1 + b_1, ..., a_n + b_n)$ belongs to W_1.
Finally, say $c \in F$ and $(a_1, ..., a_n) \in W_1$. Then $c(a_1, ..., a_n) = (ca_1, ..., ca_n)$ also belongs to W_1 since $ca_1 + ... + ca_n = c(a_1 + ... + a_n) = c0 = 0$.
Therefore, by Theorem 1.3, W_1 is a subspace of F^n.

Now consider W_2. The zero vector of F^n does not belong to W_2, since the sum of its entries is $0 + ... + 0 = 0 \neq 1$. Therefore, by Theorem 1.3, W_2 is not a subspace of F^n.

Exercise 12

Once again, we can solve the problem using Theorem 1.3.
The zero matrix O is upper triangular, since $O_{ij} = 0$ for all i, j.
Say that matrices A and B are upper triangular, so that $A_{ij} = 0 = B_{ij}$ for $i > j$. Then $(A + B)_{ij} = A_{ij} + B_{ij} = 0 + 0 = 0$ for $i > j$, which means that $A + B$ is upper triangular.
Say finally that $c \in F$ and A is an upper triangular matrix, so that $A_{ij} = 0$ for $i > j$. Then $(cA)_{ij} = aA_{ij} = c \cdot 0 = 0$ for $i > j$, which means that cA is upper triangular.
Thus upper triangular matrices form a subspace of $M_{m \times n}(F)$.

Exercise 20

Given vectors $w_1, ..., w_n \in W$ and scalars $a_1, ..., a_n \in F$, we know that $a_i w_i \in W$ for all $i = 1, ..., n$ by Theorem 1.3c. By Theorem 1.3b, we also know that the sum of any **two** vectors of W still belongs to W. Therefore, $a_1 w_1 + a_2 w_2 \in W$. This implies that $a_1 w_1 + a_2 w_2 + a_3 w_3 = (a_1 w_1 + a_2 w_2) + a_3 w_3 \in W$. Iterating this process, we have that $a_1 w_1 + ... + a_{k-1} w_{k-1} + a_k w_k = (a_1 w_1 + ... + a_{k-1} w_{k-1}) + a_k w_k \in W$ for every $k = 1, ..., n$.