1. (40pts; 10pts each) Decide whether the following statements are true or false, and provide a brief justification or counterexample (failing to do so will result in no points).

(a) Subsets of linearly dependent sets are linearly dependent.

F. $S' \subset S \subset \mathbb{R}^2$ $S = \{(1,0),(0,0)\}$ is linearly dependent, but $S' = \{(1,0)\}$ is lin. indep.

(b) For A and B two $n \times n$ matrices, $\det(A^t + B) = \det(A) + \det(B)$.

F. $\det(A+B) \neq \det(A) + \det(B)$ in general.

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$|A| = 1$, $|B| = 4$, $|A+B| = \begin{vmatrix} 3 & 1 \\ 0 & 3 \end{vmatrix} = 9 \neq 1+4$..

(c) Every system of n linear equations in n unknowns can be solved by Cramer's rule.

F. Cramer's rule requires $\det(A) \neq 0$, where A is the matrix of coefficients of the linear system.

(d) There is a unique alternating n-linear function $\delta : M_{n \times n}(R) \to R$.

F. It is unique only after specifying $\delta(I_n)$. Assigning different values to $\delta(I_n)$ corresponds to defining different functions.

2. (30pts) Use the test for diagonalization on

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

to decide whether A is diagonalizable or not and, if it is, determine its diagonal form.

$$|A - tI| = \begin{vmatrix} -t & 1 & 0 \\ 1 & -t & 0 \\ 0 & 0 & 1-t \end{vmatrix} = (1-t)(t^2-1) = $$
$$= (1-t)(t-1)(t+1)$$

$\lambda_1 = 1 \quad m_1 = 2$

$\lambda_2 = -1 \quad m_2 = 1$

$\dim E_{\lambda_2}$ must be 1, since $m_2 = 1$.

$$\dim E_{\lambda_1} = 3 - rk\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 3 - 1 = 2 = m_2.$$

Thus A is diagonalizable, with diagonal form $D = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$.

3. (30pts) Let $P_2(R)$ be the the vector space over R of polynomials $p(x)$ of degree at most two. Let D_1 be the linear functional assigning to $p(x)$ its first derivative $p'(0)$. Let D_2 be the linear functional assigning to $p(x)$ its second derivative $p''(0)$.

(a) (15pts) Prove that the set $S := \{D_1, D_2\}$ is linearly independent.

Let f_0 be the zero functional ($f_0(p(x)) = 0$ $\forall p(x) \in P_2(\mathbb{R})$).

Let $\lambda, \mu \in \mathbb{R}$ such that $\lambda D_1 + \mu D_2 = f_0$.

Then:
$$\begin{cases} (\lambda D_1 + \mu D_2)(x) = \lambda \cdot D_1(x) + \mu \cdot D_2(x) = \lambda \cdot 1 + \mu \cdot 0 \\ (\lambda D_1 + \mu D_2)(x^2) = \lambda \cdot D_1(x^2) + \mu \cdot D_2(x^2) = \\ = \lambda \cdot 0 + \mu \cdot 2 \end{cases}$$

$$\Rightarrow \begin{cases} \lambda = 0 \\ 2\mu = 0 \longrightarrow \mu = 0. \end{cases}$$

Thus S is L.I.

(b) (15pts) Find a basis S' for the dual space to $P_2(R)$ that contains S (in particular, you need to verify that the S' you find is a basis).

Let $f \in P_2(\mathbb{R})^*$ be the functional $f(a + bx + cx^2) = a$.

$f \notin \operatorname{span}\{D_1, D_2\}$; namely:

assume $\exists \lambda, \mu \in \mathbb{R}$ s.t. $f = \lambda D_1 + \mu D_2$. Then:

$$1 = f(1) = (\lambda D_1 + \mu D_2)(1) = \lambda \cdot D_1(1) + \mu \cdot D_2(1) =$$
$$= \lambda \cdot 0 + \mu \cdot 0 = 0 \quad \text{contradiction.}$$

Since S is L.I., $S' := \{f, D_1, D_2\}$ is also L.I. subset of $P_2(\mathbb{R})^*$.

Since $\dim P_2(\mathbb{R})^* = \dim P_2(\mathbb{R}) = 3$,

S' is the required basis.

4. (35pts) Let W be the subspace of 3×3 skew-symmetric real matrices (these are the real matrices such that $M^t = -M$).

Find a basis for W and compute the dimension $\dim W$ of W.

$$W = \left\{ M = (a_{ij}) \mid a_{ij} = -a_{ji} \ \forall i,j \right\} =$$

$$= \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix} \middle| \ a_{12}, a_{13}, a_{23} \in \mathbb{R} \right\}.$$

Every such M can be written as:

$$a_{12}(E^{12} - E^{21}) + a_{13}(E^{13} - E^{31}) + a_{23}(E^{23} - E^{32})$$

and the matrices in the linear comb. are lin. indep.

Thus $\dim W = 3$, since $\{E^{12} - E^{21}, E^{13} - E^{31}, E^{23} - E^{32}\}$ is a basis for W.

5. (35pts)

(Use only the methods we have learned in this course.)

Let F be a field. Let $\{e_1, e_2\}$ be the standard ordered basis in F^2 and let $\{x, y\}$ be its dual basis. Consider the ordered basis in F^2

$$\beta = \{(2,4), (-1,3)\}.$$

Find the dual basis β^* as a linear combination of the linear functionals x and y.

Let $\beta^* = \{f_1, f_2\}$.

$f_1 = ax + by$ for some $a, b \in F$.

$$\begin{cases} 1 = f_1(2,4) = 2a + 4b \rightarrow 10b = 1 \rightarrow b = \frac{1}{10} \\ 0 = f_1(-1,3) = -a + 3b \rightarrow a = 3b \rightarrow a = \frac{3}{10} \end{cases}$$

Thus $f_1 = \frac{3}{10}x + \frac{1}{10}y$.

$f_2 = ax + by$ for some $a, b \in F$.

$$\begin{cases} 0 = f_2(2,4) = 2a + 4b \rightarrow a = -2b \rightarrow a = -\frac{2}{5} \\ 1 = f_2(-1,3) = -a + 3b \rightarrow 5b = 1 \rightarrow b = \frac{1}{5} \end{cases}$$

Thus $f_2 = -\frac{2}{5}x + \frac{1}{5}y$.

6. (40pts)

Let $T : R^2 \to R^2$ be a linear map, let $\beta = \{(-3,5),(-1,5)\}$ be a basis for the domain, let $\beta' = \{(2,1),(-1,-3)\}$ be a basis for the codomain, let

$$[T]_\beta^{\beta'} = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}.$$

Prove that T is an isomorphism, and compute $T^{-1}((8,14))$ explicitly as a vector in R^2.

$$\det [T]_\beta^{\beta'} = \det \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} = 6-4 = 2 \neq 0 \Rightarrow$$

$\Rightarrow$ T is an isomorphism, and

$$[T^{-1}]_{\beta'}^{\beta} = \left([T]_\beta^{\beta'}\right)^{-1} = \frac{1}{2}\cdot\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 8 \\ 14 \end{pmatrix} = a\begin{pmatrix} 2 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ -3 \end{pmatrix} \Rightarrow \begin{cases} 6b+28-b=8 \to b=-4 \\ a=3b+14 \to a=2 \end{cases} \Rightarrow$$

$$\Rightarrow \left[\begin{pmatrix} 8 \\ 14 \end{pmatrix}\right]_{\beta'} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}.$$

$$[T^{-1}(8,14)]_\beta = [T^{-1}]_{\beta'}^{\beta} \cdot \left[\begin{pmatrix} 8 \\ 14 \end{pmatrix}\right]_{\beta'} = \frac{1}{2}\cdot\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -4 \end{pmatrix} =$$

$$= \frac{1}{2}\cdot\begin{pmatrix} 8 \\ -20 \end{pmatrix} = \begin{pmatrix} 4 \\ -10 \end{pmatrix}.$$

$$T^{-1}(8,14) = 4\cdot\begin{pmatrix} -3 \\ 5 \end{pmatrix} - 10\cdot\begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -12+10 \\ 20-50 \end{pmatrix} =$$

$$= \begin{pmatrix} -2 \\ -30 \end{pmatrix}.$$

7. (35pts) Determine the Jordan canonical form of the complex matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{bmatrix}.$$

$$|A - tI| = \begin{vmatrix} 2-t & -1 & 0 & 1 \\ 0 & 3-t & -1 & 0 \\ 0 & 1 & 1-t & 0 \\ 0 & -1 & 0 & 3-t \end{vmatrix} = (2-t)(3-t)\cdot((3-t)(1-t)+1) =$$

$$= (2-t)(3-t)(t^2-4t+4) = (2-t)(3-t)(t-2)^2.$$

$\lambda_1 = 2 \qquad m_1 = 3$

$\lambda_2 = 3 \qquad m_2 = 1$

$$\dim E_{\lambda_1} = 4 - rk\begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = 4-2=2,$$

So there are 2 blocks associated to λ_1.

$$\text{So } J = \begin{pmatrix} 3 & & & \\ & 2 & & \\ & & 2 & 1 \\ & & & 2 \end{pmatrix}$$

8. (35pts) Let A and B be similar matrices with real entries. Let $p(t) = -t^3 - 2t^2 + 3t$ be the characteristic polynomial of A. (Remember to justify your answers.)

(a) (10pts) Is B invertible?

Since A and B are similar, they have ~~the~~ the same characteristic polynomial $p(t)$.

$\det(B) = \det(B - 0 \cdot I) = p(0) = 0 \Rightarrow$

$\Rightarrow$ B is not invertible

(a matrix is invertible iff its determinant is nonzero).

(b) (10pts) Is B diagonalizable?

$p(t) = (-t)(t^2 + 2t - 3) = (-t)(t+3)(t-1)$.

B has size 3 and 3 distinct eigenvalues, thus B is diagonalizable.

(c) (15pts) Write B^4 as a linear combination of I, B, B^2.

By Cayley-Hamilton thm., $p(B) = 0$.

Thus $B^3 = 3B - 2B^2$.

$B^4 = B^3 \cdot B = (3B - 2B^2) \cdot B = 3B^2 - 2B^3 =$

$= 3B^2 - 2(3B - 2B^2) = 7B^2 - 6B$.

9. (35pts) Let V and W be n-dimensional vector spaces over a field, let $T : V \to W$ be a linear map and let β be a basis for V.

Prove that T is an isomorphism if and only if $T(\beta)$ is a basis for W.

Let $\beta = \{v_1, \dots, v_n\}$.

($\Longrightarrow$) T isomorphism $\Rightarrow$ T 1-1 $\Rightarrow$ $T(\beta)$ is L.I.

Namely:

$$0 = \sum_{i=1}^{n} a_i T(v_i) = T\left(\sum_{i=1}^{n} a_i v_i\right) \underset{\substack{\text{since}\\ N(T)=\{0\}}}{\Longrightarrow} \sum_{i=1}^{n} a_i v_i = 0 \Longrightarrow$$

$$\underset{\substack{\text{since}\\ \beta \text{ basis}}}{\Longrightarrow} a_i = 0 \ \forall i.$$

$\#T(\beta) = n = \dim W$, $T(\beta)$ L.I. $\Rightarrow$ $T(\beta)$ basis.

($\Longleftarrow$) Assume $T(\beta)$ is a basis.

$\dim V = \dim W \Longrightarrow$ by the dimension thm., to prove T isomorphism is enough to show T 1-1 (i.e. $N(T) = \{0\}$).

$v \in N(T)$

$v = \sum_{i=1}^{n} a_i v_i \Longrightarrow T(v) = 0 \Longrightarrow$

$$\Longrightarrow 0 = T(v) = \sum_{i=1}^{n} a_i T(v_i) \Longrightarrow a_i = 0 \ \forall i$$

since $T(\beta)$ is a basis.

Thus T 1-1.

10. (35pts) (This exercise has three parts a), b) and c), which is on the next page.)

Let F be an algebraically closed field. All matrices in this problem are matrices in $M_{2\times 2}(F)$.

(a) (10pts)

Let $0 \neq \lambda \in F$. Find a square root matrix B of the 2×2 Jordan block $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, that is find a matrix B such that $B^2 = A$.

Assume $B = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $B^2 = \begin{pmatrix} a^2 & ab+bc \\ 0 & c^2 \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

Since F is alg. closed and $\lambda \neq 0$, $\exists \sqrt{\lambda}$ and $\sqrt{\lambda} \neq 0$.

Thus $a = \sqrt{\lambda}$, $c = \sqrt{\lambda}$, $b \cdot (\sqrt{\lambda} + \sqrt{\lambda}) = 1 \Rightarrow b = \frac{1}{2\sqrt{\lambda}}$ is a solution; i.e.,

$\begin{pmatrix} \sqrt{\lambda} & \frac{1}{2\sqrt{\lambda}} \\ 0 & \sqrt{\lambda} \end{pmatrix}$ is a square root.

(b) (10pts) Prove that a matrix C admits a square root if and only iff every matrix D similar to C admits a square root.

D similar to $C \Rightarrow \exists Q$ invertible s.t. $D = Q^{-1}CQ$. If B is a square root of C, then $D = Q^{-1}B^2Q = (Q^{-1}BQ)^2$, i.e., $Q^{-1}BQ$ is a square root of D.

If D has a square root, $C = QDQ^{-1}$ and thus the same argument applies.

(If $B^2 = D$, QBQ^{-1} is a square root of C).

(c) (15pts) Prove that every invertible matrix E admits a square root (Hint: consider the possible Jordan canonical forms for E and then use (a) and (b).)

The possible Jordan forms for E are:

1) $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with $\lambda_1, \lambda_2 \in F$ possibly equal.

2) $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ with $\lambda \in F \setminus \{0\}$ (otherwise $\det \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \lambda^2 = 0 \Rightarrow \det(E) = 0$ $\Rightarrow E$ not invertible, contrary to the assumption).

(similar matrices have the same determinant, and $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, E are similar).

$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ has square root $\begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix}$ (since F is algebraically closed).

$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\lambda \neq 0$ has a square root by (a).

Since these are the possible Jordan forms of E and hence similar to E, E has a square root by (b).