MAT 142 Spring 2003
MIDTERM II

!!! WRITE YOUR NAME, STUDENT ID BELOW !!!

NAME : ID :

THERE ARE 6 PROBLEMS. THEY DO NOT HAVE EQUAL VALUE.
SHOW YOUR WORK!!!

1	40	
2	40	
3	40	
4	40	
5	40	
6	50	
Total	250	

1. [**40 points**] Use partial fractions to integrate

$$\int \frac{y+3}{(y^2+1)(y-1)}\,dy$$

$\frac{y+3}{(y^2+1)(y-1)} = \frac{Ay+B}{y^2+1} + \frac{C}{y-1}$
$y+3 = (Ay+B)(y-1)+C(y^2+1)$ Plugging in $y = 1$, we get $4 = C \cdot 2$. So $C = 2$.
Plugging in $y = 0$, we get $3 = -B + C$. So $B = C - 3 = -1$.
Comparing the highest order terms, we get $0 = A + C$. So $A = -C = -2$.

$$\begin{aligned}\int \frac{y+3}{(y^2+1)(y-1)}\,dy &= \int \frac{-2y-1}{y^2+1} + \frac{2}{y-1}\,dy \\ &= \int -\frac{2y}{y^2+1} - \frac{1}{y^2+1} + \frac{2}{y-1}\,dy \\ &= -\ln(y^2+1) - \arctan y + 2\ln|y-1| + C\end{aligned}$$

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2. [**40 points**] Solve the initial value problem

$$\sqrt{9+x^2}\,\frac{dy}{dx} = 1 \qquad y(0) = 0$$

Dividing each side by $\sqrt{9+x^2}$ we get:

$$\frac{dy}{dx} = \frac{1}{\sqrt{9+x^2}}$$

Integrating both side by x and using substitution $x = 3\tan\theta$:

$$\begin{aligned} y = \int \frac{1}{\sqrt{9+x^2}}\,dx &= \int \frac{1}{|3\sec\theta|} 3\sec^2\theta\,d\theta \\ &= \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C \\ = \ln|\sqrt{1+\tan^2\theta} + \tan\theta| + C &= \ln|\sqrt{1+\frac{x^2}{9}} + \frac{x}{3}| + C \end{aligned}$$

Since $y(0) = 0$, $\ln\sqrt{1+0} + 0 + C = 0$. So $C = 0$.

We now have $y(x) = \ln|\sqrt{1+\frac{x^2}{9}} + \frac{x}{3}|$.

3. Evaluate.

a) **[20 points]**

$$\lim_{x\to 0}(\cos x)^{\csc x}$$

$$\lim_{x\to 0}(\cos x)^{\csc x} = \lim_{x\to 0} e^{\csc x \ln(\cos x)}$$
$$= \lim_{x\to 0} e^{\frac{\ln(\cos x)}{\sin x}} = \lim_{x\to 0} e^{\frac{\frac{-\sin x}{\cos x}}{\cos x}}$$
$$= \lim_{x\to 0} e^{-\frac{tan\, x}{\cos x}} = e^{-\frac{0}{\cos 0}} = e^0 = 1$$

b) **[20 points]**

$$\lim_{x\to 0^+}(\frac{1}{x} - \frac{1}{\sin x})$$

$$\lim_{x\to 0^+}(\frac{1}{x} - \frac{1}{\sin x}) = \lim_{x\to 0^+}(\frac{\sin x - x}{x \sin x})$$
$$= \lim_{x\to 0^+}(\frac{\cos x - 1}{\sin x + x\cos x}) = \lim_{x\to 0^+}(\frac{-\sin x}{\cos x + \cos x - x \sin x})$$
$$= \frac{0}{2} = 0$$

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4. [**40 points**] Let

$$A = \lim_{t\to\infty} \frac{e^t + 2}{e^t + 5}$$

and

$$B = \lim_{t\to 0} \frac{e^t + 2}{e^t + 5}$$

a) In each case, are you allowed to use L'Hopital's rule? Explain.

A: Yes. This is a limit of $\frac{\infty}{\infty}$ type. Both denominator and numerator are differentiable on $[0, \infty)$ and the limit of $\lim_{t\to\infty} \frac{e^t}{e^t}$ exists.
B: No. At 0, both denominator and numerator have nonzero and finite value.

b) Evaluate A and B.

$$A = \lim_{t\to\infty} \frac{e^t + 2}{e^t + 5} = \lim_{t\to\infty} \frac{e^t}{e^t} = 1$$

$$B = \lim_{t\to 0} \frac{e^t + 2}{e^t + 5} = \frac{e^0 + 2}{e^0 + 5} = \frac{3}{6} = 0.5$$

5. [**40 points**] Evaluate or prove that the following improper integral diverges

$$\int_{-\infty}^{0} x\, e^{x}\, dx.$$

$$\int_{-\infty}^{0} x\, e^{x}\, dx. = \lim_{t\to\infty} \int_{-t}^{0} x\, e^{x}\, dx.$$

$$= \lim_{t\to\infty} xe^{x}|_{-t}^{0} \; - \; \int_{-t}^{0} e^{x}\, dx.$$

$$= \lim_{t\to\infty} te^{-t} \; - \; e^{x}|_{-t}^{0}$$

$$= \lim_{t\to\infty} \frac{1+t-e^{t}}{e^{t}} = \lim_{t\to\infty} \frac{1-e^{t}}{e^{t}}$$

$$= \lim_{t\to\infty} \frac{-e^{t}}{e^{t}} = -1$$

6.

(a) [**30 points**] Determine whether or not the sequence

$$a_n = (-1)^n \frac{3 + e^{-n}}{1 + e^{-n}} + \frac{1}{n}$$

converges. If it does compute the limit. (Do not use ϵ, N; but justify your answer).

Suppose that a_n converges. Since we know that $\lim_{n\to\infty} \frac{1}{n} = 0$, $a_n - \frac{1}{n} = (-1)^n \frac{3+e^{-n}}{1+e^{-n}}$ has to converge by difference rule. But this sequence diverges since the even terms converge to 3 while odd terms converge to -3. Hence this is a contradiction. So a_n diverges.

(b) [**20 points**] Determine whether or not the limit below exists and if it does compute it.

$$\lim_{n\to\infty} \frac{\ln n}{5^{n^2}}$$

The numerator and the denominator of the function

$$\frac{\ln x}{5^{x^2}}$$

are differentiable and this function agrees with $\lim_{n\to\infty} \frac{\ln n}{5^{n^2}}$ on all integers.

So we may calculate the limit of the function using L'Hospital's Law.

$$\lim_{x\to\infty} \frac{\ln x}{5^{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{2x \cdot 5^{x^2} \ln 5}$$

$$= \lim_{x\to\infty} \frac{1}{2x^2 \cdot 5^{x^2} \ln 5} = 0$$

. So the limit of $\lim_{n\to\infty} \frac{\ln n}{5^{n^2}}$ exists and is zero.