MAT 142 SPRING 2003
MIDTERM I

!!! WRITE YOUR NAME, SUNY ID N. AND SECTION BELOW !!!

NAME :

SUNY ID N. :

SECTION :

CHECK THAT THERE ARE ??? PROBLEMS. THEY DO NOT HAVE EQUAL VALUE. SHOW YOUR WORK!!!

1	40	
2	40	
3	50	
4	40	
5	80	
Total	250	

1. [**40 points**] Find the center of mass of a thin plate covering the region bounded by the parabola $y = 2x^2$ and by the line $y = 2x$ if the plate's density at the point (x, y) is $\delta(x) = 10x$.

$2x^2 = 2x$
$x(x-1) = 0$
The two graphs meet at (0,0) and (1,1).
Using vertical strips, $(\tilde{x}, \tilde{y}) = (x, x^2 + x)$.

$$M_x = \int_0^1 \tilde{y}\delta(x)(2x - 2x^2)\,dx = \int_0^1 (x^2 + x)10x(2x - 2x^2)\,dx$$

$$= 20\int_0^1 x(-x^4 + x^2)\,dx = 20\int_0^1 -x^5 + x^3\,dx = 20(-\frac{1}{6}x^6 + \frac{1}{4}x^4)|_0^1 = \frac{5}{3}.$$

$$M_y = \int_0^1 \tilde{x}\delta(x)(2x - 2x^2)\,dx = \int_0^1 (x)10x(2x - 2x^2)\,dx$$

$$= 20\int_0^1 x^3 - x^4\,dx = 20(\frac{1}{4}x^4 - \frac{1}{5}x^5)|_0^1 = 5 - 4 = 1.$$

$$M = \int_0^1 \delta(x)(2x - 2x^2)\,dx = \int_0^1 10x(2x - 2x^2)\,dx$$

$$= 20\int_0^1 x^2 - x^3\,dx = 20(\frac{1}{3}x^3 - \frac{1}{4}x^4)|_0^1 = \frac{5}{3}.$$

So the answer is:

$$(\bar{x}, \bar{y}) = (\frac{M_y}{M}, \frac{M_x}{M}) = (\frac{3}{5}, 1)$$

2. [**40 points**] An isosceles triangular plate with basis $5\,ft$ and height $10\,ft$ stands with its base at the bottom of a pool of water $16\,ft$ deep. Find the fluid force on one side of the plate. (Density of water: $62.4\,lb/ft^3$.)

$L(0) = 5$,$L(10) = 0$, and since $L(y)$ is linear, $L(y) = 5 - \frac{y}{2}$.
So the total force acting on this plate is:

$$\int_0^{10} 62.4(16-y)(5-\frac{y}{2})\,dy = 31.2\int_0^{10} y^2-26y+160\,dy = 31.2(\frac{1}{3}y^3-13y^2+160y)|_0^{10}$$
$$= 10400 - 40560 + 49920 = 19760\ (lb).$$

3. [**50 points**] A tank contains 200 *gal* of pure palm oil. Olive oil is being added to this tank at a fixed rate of 2 gal/min while the well mixed mixture of oils in the tank is being drained out at a rate of 2 gal/min.
After how many minutes will the tank contain 50% olive oil?

rate in of olive oil $= 2\ gal/min$
Since rate of oil being lost is same as rate of oil being gained, volume of the total oil in the tank is always 200, so
rate out of olive oil $= \frac{s(t)}{200} \cdot$ (rate out) $= s(t)/100$.
($s(t)$ is the amount of olive oil in the tank at time t.)
So we have a differential equation :

$$\frac{ds}{dt} = 2 - \frac{s}{100}.$$

Define $v(t) = e^{\int \frac{1}{100} dt} = e^{.01t}$.
Then the solution is $s(t) = \frac{1}{v(t)} \int v(t)Q(t)\, dt = e^{-.01t}(200e^{.01t} + C) = 200 + Ce^{-.01t}$.
Since initially there was no olive oil in the tank, $s(0) = 0$. Hence $C = -200$.
Now 50% olive oil means $s = 100$. So solving the equation:

$$100 = 200 - 200e^{-.01t}$$

we get $t = 100 \ln 2\ (min)$.

4. [**40 points**] Prove the following:

a) $\cosh^2 x - \sinh^2 x = 1$

$\cosh^2 x - \sinh^2 x = (\frac{e^x+e^{-x}}{2})^2 - (\frac{e^x-e^{-x}}{2})^2 = \frac{e^{2x}+2e^xe^{-x}+e^{-2x}}{4} - \frac{e^{2x}-2e^xe^{-x}+e^{-2x}}{4}$
$= \frac{4e^xe^{-x}}{4} = e^xe^{-x} = e^0 = 1.$

b) $\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$

Let $y = \cosh^{-1} x$. Then applying cosh both sides, we get :

$$\cosh y = x.$$

Using implicit differentiation we obtain:

$$\sinh y \cdot y' = 1.$$

We now have the following:

$$\frac{d}{dx}\cosh^{-1} x = y' = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2-1}}.$$

(For the third equality, we are using part a).)

5. [**80 points**] Integrate the following four expressions:

a) $\int \frac{1}{\sqrt{x^2-2x}}\,dx$

$$\int \frac{1}{\sqrt{x^2-2x}}\,dx = \int \frac{1}{\sqrt{(x-1)^2-1}}\,dx.$$

Change of variables : $x-1=\cosh u,\ dx=\sinh u\,du.$

$$= \int \frac{1}{\sinh u}\,\sinh u\,du = \int 1\,du = u + C = \cosh^{-1}(x-1)+C.$$

b)$\int (\tan x + \cot x)^2\,dx$

$$\int (\tan x+\cot x)^2\,dx = \int (\tan^2 x+2\tan x\cot x+cot^2 x)\,dx = \int (\sec^2 x-1)+2+(\csc^2 x-1)\,dx$$

$$= \int \sec^2 x + \csc^2 x\,dx = \tan x - \cot x + C.$$

c) $\int \frac{x^3+1}{x^2+1}\,dx$

$$\int \frac{x^3+1}{x^2+1}\,dx = \int x+\frac{-x+1}{x^2+1}\,dx = \int x-\frac{1}{2}\cdot\frac{2x}{x^2+1}+\frac{1}{x^2+1} = \frac{1}{2}x^2-\frac{1}{2}\ln(x^2+1)+\tan^{-1}x+C.$$

d) $\int_0^{2\pi}\sqrt{1+\cos x}\,dx$

$$\int_0^{2\pi}\sqrt{1+\cos x}\,dx = \int_0^{2\pi}\sqrt{2\cdot\frac{1+\cos x}{2}}\,dx = \int_0^{2\pi}\sqrt{2\cdot\cos^2(\frac{x}{2})}\,dx$$

$$= \sqrt{2}\int_0^{2\pi}|\cos(\frac{x}{2})|\,dx = 2\sqrt{2}\int_0^{\pi}\cos(\frac{x}{2})\,dx = 4\sqrt{2}\sin(\frac{x}{2})|_0^{\pi} = 4\sqrt{2}.$$