

## Rational, algebraic and transcendental numbers

**DEFINITION.** A real number  $r$  is said to be *rational* if it is the ratio  $m/n$  of two integers.

**EXAMPLE.**  $2/3$ ,  $(-6)/5$ .

**Definition.** A real number  $r$  is said to be *algebraic* if it is a solution to a polynomial equation where the coefficients of the polynomial are integers.

**EXAMPLE.**  $\sqrt{2}$  is algebraic because it is a solution to  $p(x) = -2x^2 + 4 = 0$ .

**FACT.** Every rational number is algebraic. Show it! (Hint: a linear polynomial will do.)

**DEFINITION.** A real number  $r$  is said to be *transcendental* if it is not algebraic.

We have that rational numbers are algebraic, algebraic numbers are not necessarily rational (see below), and irrational numbers can be either algebraic or transcendental, but not both!

The numbers  $e$  and  $\pi$  are transcendental. This is not easy to prove.

It is not known whether  $\pi^e$  is transcendental.

In a precise sense: “most real numbers” are transcendental; however there are enough rational numbers to allow us to compute!

**PROBLEM.** Show that  $\sqrt{2}$  is not rational. (Hint: argue by contradiction; choose the integers  $m$  and  $n$  to have no common factors; use that all positive integers can be factored as a product of prime numbers)

**PROBLEM.** What can you say about the rationality of  $\sqrt{t}$  when  $t$  is a positive integer?