PROOFS USING (ϵ, N)

Q. Show that
$$\lim_{n\to\infty} -2 - \frac{3}{n^5} = -2.$$

Proof.

Fix $\epsilon > 0$.

Consider the inequality*
$$|(-2 - \frac{3}{n^5}) - (-2)| < \epsilon.$$

We simplify
$$|\frac{3}{n^5}| < \epsilon.$$

We solve** for n!!!:
$$|\frac{3}{\epsilon}| < n^5$$

or
$$n > (\frac{3}{\epsilon})^{\frac{1}{5}}.$$

The solution* is** $N = N(\epsilon) =$ **any integer bigger than** $(\frac{3}{\epsilon})^{\frac{1}{5}}$.

Remarks.

$*$ *(Here: $a_n = (-2 - \frac{3}{n^5})$, and $L = -2$.)*

$**$ *(We are looking for $N = N(\epsilon)$ such that the inequality is true for every $n > N$.)*

$***$ *For every $n > N$ the inequality $n > N > (\frac{3}{\epsilon})^{\frac{1}{5}}$ is true, so the inequality $n > (\frac{3}{\epsilon})^{\frac{1}{5}}$ is true, so the inequality $\frac{3^{\frac{1}{5}}}{n} < \epsilon^{\frac{1}{5}}$ is true, so the inequality $\frac{3}{n^5} < \epsilon$ is true for every $n > N$ and we are done.*

Q. Show that the sequence $a_n = n$ diverges.

Proof. By contradiction assume* that

$$\lim_{n\to\infty} n = L$$

for some number L.

This means that if we choose any ϵ, for example $\epsilon = 1$, then there is a positive integer N such that

$$|n - L| < 1$$

for every $n > N$.

Let n be any integer bigger than N and bigger than $|L| + 1$.

The inequality $|n - L| < 1$ is then violated for this n which is bigger than N and we have reached a contradiction**.

Remarks. *∗This assumption, which we will prove to be false, means that : for every $\epsilon > 0$, there is a positive integer N such that for every $n > N$ we have that $|a_n - L| < \epsilon$.*

∗∗ It contradicts what above: it is not true that for every ϵ there is that N, in fact we hand-picked $\epsilon = 1$ and showed that no matter how you choose N, the inequality $|a_n - L| < \epsilon$. is violated for some $n > N$.