12) Prove by induction on n that 3 divides $4^n + 5$.

First we must check the $n = 1$ case:

$$4^1 + 5 = 4 + 5 = 9 = 3 \cdot 3$$

So this shows that, by definition, 3 divides 9, and so we're done with the $n = 1$ case. Now we must prove the inductive step. This means we must assume that

$$4^k + 5 = 3p$$

and show that this implies that

$$4^{k+1} + 5 = 3q$$

for some positive integer q. So now we say that:

$$\begin{aligned} 4^{k+1} + 5 &= 4^{k+1} + 20 - 15 \\ &= 4 \cdot 4^k + 4 \cdot 5 - 15 \\ &= 4(4^k + 5) - 15 \\ &= 4(3p) - 3 \cdot 5 \\ &= 3(4p - 5) \end{aligned}$$

Since $4p - 5 > 1$, it is a positive integer, and thus can be our q, and we are done.

13) Prove by induction on n that $n! > 2^n$ for all integers n such that $n \geq 4$.

Here our first case to check is $n = 4$:

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 > 32 = 2^4$$

So we're good here. Now for the inductive step, we must assume that

$$k! > 2^k$$

and show that this assumption implies that:

$$(k + 1)! > 2^{k+1}$$

So we begin:

$$\begin{aligned}(k+1)! &= (k+1)k! \\ &> (k+1)2^k \\ &> 2\cdot 2^k \\ &= 2^{k+1}\end{aligned}$$

And we are finished.

14) Prove Bernoulli's inequality: $(1+x)^n \geq 1+nx$ for all non-negative integers n and all real numbers $x \geq -1$.

Here, throughout the proof, every statement we make must hold for any value x may take so long as $x \geq 1$. Also, since the statement is about all non-negative integers, this means the first case we must check is $n=0$, so let's check:

$$\begin{aligned}(1+x)^0 &= 1 \\ &= 1+0\cdot x\end{aligned}$$

so that was sure easy! Next we must prove the induction step, so as usual we need to assume that $(1+x)^k \geq 1+kx$ is indeed true, and show that this implies that $(1+x)^{k+1} \geq 1+(k+1)x$ must also be true. So let's go!

$$\begin{aligned}(1+x)^{k+1} &= (1+x)(1+x)^k \\ &\geq (1+x)(1+kx) \\ &= 1+(k+1)x+x^2 \\ &\geq 1+(k+1)x\end{aligned}$$

and once again we're finished. But where did we need to assume that $x \geq -1$? Note the use of one of the order axioms in obtaining the very first inequality. If $1+x<0$, then the inequality would have been reversed, and we'd have been up the creek.

16) Prove by induction on n that $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.

First the $n=1$ case:

$$\begin{aligned}\sum_{i=1}^{1} \frac{1}{i(i+1)} &= \frac{1}{1(1+1)} \\ &= \frac{1}{1+1}\end{aligned}$$

Man these base cases are easy! Awesome! Alright, now for the hard part. We must assume that

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$

and show that this assumption implies that:

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{(k+1)+1}$$

And away we go:

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2}
\end{aligned}$$

and once again we're finished.

17) For a positive integer n the number a_n is defined inductively by

$$\begin{aligned}
a_1 &= 1 \\
a_{k+1} &= \frac{6a_k+5}{a_k+2}
\end{aligned}$$

Prove by induction on n that for all positive integers, $a_n > 0$ and $a_n < 5$.

Ok, here the $n = 1$ case is obvious. So we will proceed with the induction step. So suppose that we have $0 < a_k < 5$ and let's show that this implies $0 < a_{k+1} < 5$. In this proof, I worked backwards on scratch paper, and then typed it out:

$$\begin{aligned} a_k &< 5 \\ \Rightarrow 6a_k + 5 &< 5a_k + 10 \\ \Rightarrow \frac{6a_k + 5}{a_k + 2} &< 5 \end{aligned}$$

and since the left hand side of the last line is the very definition of a_{k+1}, we're done with the first part. To show $a_{k+1} > 0$, we can work backwards again:

$$\begin{aligned} 0 &< a_k \\ \Rightarrow 0 < 6a_k + 5 &\text{ and } 0 < a_k + 2 \\ \Rightarrow 0 &< \frac{6a_k + 5}{a_k + 2} \end{aligned}$$

21) Suppose that x is a real number such that $x + \frac{1}{x}$ is an integer. Prove by induction on n that $x^n + \frac{1}{x^n}$ is an integer for all positive n.

Again I'm going to skip the $n = 1$ case. The book gives us a really nice hint for this one, so let's take advantage of it:

$$(x^k + \frac{1}{x^k})(x + \frac{1}{x}) = x^{k+1} + \frac{1}{x^{k+1}} + x^{k-1} + \frac{1}{x^{k-1}}$$

Now, if we use the *strong* induction principle, i.e. assume that $x^j + \frac{1}{x^j}$ is an integer for all $j \leq k$, we can use this to show that $x^{k+1} + \frac{1}{x^{k+1}}$ must be an integer, as well. Since we're assuming that $x^k + \frac{1}{x^k}$, $x + \frac{1}{x}$, and $x^{k-1} + \frac{1}{x^{k-1}}$ are all integers, most of the terms above are actually integers! Now it looks like we're on to something. From the above equality, we can get:

$$(x^k + \frac{1}{x^k})(x + \frac{1}{x}) - (x^{k-1} + \frac{1}{x^{k-1}}) = x^{k+1} + \frac{1}{x^{k+1}}$$

and by the previous comments, we see that the left hand side is of the form $pq - r$ with $p = x^k + \frac{1}{x^k}$, $q = x + \frac{1}{x}$, and $r = x^{k-1} + \frac{1}{x^{k-1}}$. Since p, q, and r are all integers, so is $pq - r$, and we're done.