Classifying Spaces and Representability Theorems

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We will use the following notation for some common categories:

CW - The category of CW-complexes and continuous functions

hCW - The category of CW-complexes and homotopy classes of continuous functions

hCW$_*$ - The category of based CW-complexes and based homotopy classes of base point preserving continuous functions

Set - The category of sets and functions

Set$_*$ - The category of based sets and base point preserving functions

1 The Brown Representability Theorem

A contravariant functor $F : \mathscr{C} \to \mathbf{Set}$ is said to be representable if there exists an object X in $\mathscr{C}$ (called a classifying object) and a natural isomorphism

$$\eta : \hom_{\mathscr{C}}(-, X) \longrightarrow F$$

Examples:

1. Clearly $\hom(-, X)$ is a representable functor on any category $\mathscr{C}$.
2. Singular cohomology $H^n(-; G)$ on **hCW** is representable since

   $$H^n(X; G) \cong [X, K(G, n)]$$

   where $K(G, n)$ is the n-th Eilenberg-MacLane space of G.
3. For a topological group G let us define a functor $k_G : \mathbf{hCW}_* \to \mathbf{Set}_*$ by

   $$k_G(X) = \{\text{isomorphism classes of principal } G\text{-bundles over } X\}$$

   For a morphism $X \xrightarrow{[f]} Y$ the induced morphism $k_G([f]) : k_G(Y) \to k_G(X)$ is given by

   $$k_G([f])(\xi) = f^*\xi$$

   As we will discuss later, it turns out that there is a space BG such that

   $$k_G(-) \simeq [-, (BG, *)]$$

When $\mathscr{C} = \mathbf{hCW}_*$ we have necessary and sufficient conditions for a contravariant functor $F : \mathbf{hCW}_* \to \mathbf{Set}_*$ to be representable. This is the content of the Brown Representability Theorem. Before stating the theorem, we define two properties the functor F might have.

Let $\{X_\alpha\}$ be a collection of spaces, and let $i_\beta : X_\beta \to \vee_\alpha X_\alpha$ be the natural inclusions (recall that the coproduct in $\mathbf{hCW}_*$ is the wedge construction, so these are the maps provided by the coproduct). We say that F satisfies the wedge axiom if for any collection of spaces $\{X_\alpha\}$, the map $i : F(\vee_\alpha X_\alpha) \to \prod_\alpha F(X_\alpha)$ induced by the maps $F(i_\beta) : F(\vee_\alpha X_\alpha) \to F(X_\beta)$ is bijective.

Now let $(X; A, B)$ be a CW-triad, i.e. A and B are subcomplexes such that $X = A \cup B$. Let j_A, j_B denote the inclusions of $A \cap B$ into the respective spaces, and let i_A, i_B denote the inclusions of the respective spaces into X. Then we say that F satisfies the Mayer-Vietoris axiom if for any CW-triad $(X; A, B)$ and for any $x \in F(A), y \in F(B)$ such that

$$F(j_A)(x) = F(j_B)(y) \in F(A \cap B)$$

there exists an element $z \in F(X)$ such that

$$F(i_A)(z) = x \qquad F(i_B)(z) = y$$

Notice that it is exactly this property that would allow us to conclude that im $\subset$ ker in the middle group of the Mayer-Vietoris sequence:

$$\cdots \longleftarrow H^k(A \cap B) \longleftarrow H^k(A) \oplus H^k(B) \longleftarrow H^k(X) \longleftarrow \cdots$$

We can now state the Brown Representability Theorem:

Lemma 1.1 (Brown)**.** *A contravariant functor F : $\boldsymbol{hCW}_* \to \boldsymbol{Set}_*$ is representable if and only if F satisfies the wedge axiom and the Mayer-Vietoris axiom.*

The idea of the proof is to construct a CW-complex (Y, y_0) and element $u \in F(Y)$ such that the natural transformation

$$T_u : [-, (Y, y_0)] \longrightarrow F(-)$$
$$T_u(X)([f]) = F([f])(u)$$

is actually an equivalence. The necessity of F to satisfiy the wedge and Mayer-Vietoris axioms is easy enough to prove. First we recall some basic constructions. For based spaces $(X, x_0), (Y, y_0)$ the smash product is defined to be

$$X \wedge Y = \frac{X \times Y}{X \vee Y} = \frac{X \times Y}{\{x_0\} \times Y \cup X \times \{y_0\}}$$

This is a based space by taking the base point to be the image of $X \vee Y$ under the quotient map. For an unbased space X let X^+ denote X union a disjoint point, which is then considered to be the base point of X^+. This seems to be a nice candidate for the product in the category of based spaces, but is not the product; the regular topological product is still the categorical product. However, the smash product is very useful due to the adjoint relation

$$[(X \wedge Y, *), (Z, z_0)] \cong [(X, x_0), (Y, y_0)^{(Z, z_0)}] \tag{1.1}$$

It is easy to see that $X \wedge I^+ \simeq (X \times I)/(\{x_0\} \times I)$, so we see that a based homopty $H : X \times I \to Y$ between maps $X \to Y$ descends to a map $H : X \wedge I^+ \to Y$. Conversely, any based map $H : X \wedge I^+ \to Y$ is also a based homotopy when considered as a map $H : X \times I \to Y$.

Lemma 1.2. *The functor $[-,(Y,y_0)]$ satisfies the wedge axiom and the Mayer-Vietoris axiom.*

Proof.

W.) The maps $i_\beta^* : [\vee_\alpha X_\alpha] \to [X_\alpha, Y]$ induce a map $i : [\vee_\alpha X_\alpha] \to \prod_\alpha [X_\alpha, Y]$ as noted before. Let $\{[f_\alpha]\} \in \prod_\alpha [X_\alpha, Y]$, then we can fit the f_α's together to form a map $f : \vee_\alpha X_\alpha \to Y$ such that $f \circ i_\alpha = f_\alpha$. If we change an f_α by a homotopy, then f also changes by a homotopy, hence we have $i([f]) = \{[f_\alpha]\}$, and so i is surjective.

Suppose $[f],[g] \in [\vee_\alpha X_\alpha, Y]$ such that $i([f]) = i([g])$. If we define $f_\alpha = f \circ i_\alpha$ and $g_\alpha = g \circ i_\alpha$, then we have $f_\alpha \simeq g_\alpha$, so let $H^\alpha : X_\alpha \wedge I^+ \to Y$ be the homotopies. These fit together to give us a map $H : \vee_\alpha (X_\alpha \wedge I^+) \to Y$ such that $H|_{X_\alpha \times 0} = f_\alpha$ and $H|_{X_\alpha \times 1} = g_\alpha$. But, since $\vee_\alpha (X_\alpha \wedge I^+) \cong (\vee_\alpha X_\alpha) \wedge I^+$, we have that H is a map $(\vee_\alpha X_\alpha) \times I^+ \to Y$ such that $H_0 = f$ and $H_1 = g$, hence $[f] = [g]$, and so i is injective.

MV.) Let $(X; A, B)$ be a CW-triad, and let $[f] \in [(A, x_0), (Y, y_0)]$ and $[g] \in [(B, x_0), (Y, y_0)]$ such that $f|_{A\cap B} \simeq g|_{A \cap B}$. Let $\widetilde{H} : (A \cap B) \wedge I^+ \to Y$ be a homotopy between f and g. Since $A \cap B \hookrightarrow A$ is a cofibration we can extend this homotopy to $H : A \wedge I^+ \to Y$ such that $H_0 = f$. Let $\widetilde{f} = H_1$, then we see that $[f] = [\widetilde{f}] \in [(A, a_0), (Y, y_0)]$, but now $\widetilde{f}|_{A\cap B} = g|_{A\cap B}$ (not just homotopic, but equal). Now we can easily extend this to a map $h : (X, x_0) \to (Y, y_0)$ such that $h|_A = \widetilde{f}$ and $h|_B = g$, and this verifies MV.

□

We will say that an element $u \in F(Y)$ is n-universal if $T_u(S^q) : [(S^q, *), (Y, y_0)] \to F(S^q)$ is an isomorphism for $q < n$ and an epimorphism for $q = n$ (recall that T_u is the natural transformation defined earlier). This element is called universal if it is n-universal for all n.

Theorem 1.1. *If $f : (X, x_0) \to (Y, y_0)$ is a morphism in $\boldsymbol{CW}_*$, and $u \in F(X), v \in F(Y)$ are universal elements such that $F(f)(v) = u$, then*

$$f_* : \pi_*(X, x_0) \longrightarrow \pi_*(Y, y_0)$$

is an isomorphism.

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} \pi_q(X,x_0) = [(S^q,*),(X,x_0)] & \xrightarrow{\quad f_* \quad} & [(S^q,*),(Y,y_0)] = \pi_q(Y,y_0) \\ & {\scriptstyle T_u(S^q)} \searrow \quad \swarrow {\scriptstyle T_v(S^q)} & \\ & F(S^q) & \end{array}$$

To see this let $[g] \in [(S^q, *), (X, x_0)]$, then $f_*([g]) = [f \circ g]$ and

$$T_v(S^q)(f \circ g) = F(f \circ g)(v) = F(g) \circ F(f)(v) = F(g)(u) = T_u(S^q)(g)$$

By assumption we have $T_u(S^q)$ and $T_v(S^q)$ are bijective, so we must have that f_* is bijective, hence it is an isomorphism. □

Most of the steps of the proof of the Brown Representability Theorem come from trying to construct universal elements, and the proofs of the lemmas that lead to such a construction are very similar to the proof of the Whitehead's theorem. We only state this theorem for now:

Lemma 1.3. *For any contravariant functor* $F : \boldsymbol{hCW}_* \to \boldsymbol{Set}_*$ *there exists a CW-complex* (Y, y_0) *and universal element* $u \in F(Y)$*. In fact, with a choice of* (Y, y_0) *and universal* $u \in F(Y)$*, the natural transformation* $T_u : [-, (Y, y_0)] \to F$ *is an equivalence.*

The following lemma is useful for when trying to make the construction of classifying spaces into a functorial construction.

Lemma 1.4. *Let* $F, F' : \boldsymbol{hCW}_* \to \boldsymbol{Set}_*$ *be two contravariant functors with classifying spaces* $(Y, y_0), (Y', y_0')$ *and universal elements* u, u' *respectively. If* $T : F \to F'$ *is a natural transformation, then there is a map* $f : (Y, y_0) \to (Y', y_0')$*, unique up to homotopy, such that the diagram*

$$\begin{array}{ccc} [(X, x_0), (Y, y_0)] & \xrightarrow{f_*} & [(X, x_0), (Y', y_0')] \\ {\scriptstyle T_u(X)}\downarrow & & \downarrow{\scriptstyle T_{u'}(X)} \\ F(X) & \xrightarrow[T(X)]{} & F'(X) \end{array}$$

commutes for all $(X, x_0) \in \boldsymbol{hCW}_*$.

Corollary 1.1. *The classifying space of* F *is unique up to homotopy equivalence.*

Proof. Suppose F has two classifying spaces (Y, y_0) and (Y', y_0') with universal elements u, u' respectively. If we let $T : F \to F$ be the identity natural transformation, then 1.4 gives us a map $f : (Y, y_0) \to (Y', y_0')$ and commutative diagram (with $X = S^q$):

$$\begin{array}{ccc} \pi_q(Y, y_0) = [(S^q, *), (Y, y_0)] & \xrightarrow{f_*} & [(S^q, *), (Y', y_0')] = \pi_q(Y', y_0') \\ {\scriptstyle T_u(S^q)}\downarrow & & \downarrow{\scriptstyle T_{u'}(S^q)} \\ F(S^q) & \xrightarrow[T(S^q) = \mathrm{id}_{F(S^q)}]{} & F(S^q) \end{array}$$

Since $T_u(S^q)$ and $T_{u'}(S^q)$ are bijective (by the universality of u and u'), we see that f_* must be bijective, and hence an isomorphism. By Whitehead's theorem we have that f is a homotopy equivalence. □

2 Universal Bundles

We now want to apply these ideas to the theory of vector bundles. Before we can do this we recall some basic facts from the theory of fiber bundles. Let $\xi : F \to E \xrightarrow{\pi} B$ be a fiber bundle with structural group $G \subset \mathrm{Diff}(F)$. Then we have an atlas $U_\alpha \subset B$, $\varphi_\alpha : U_\alpha \times F \xrightarrow{\sim} \pi^{-1}(U_\alpha)$. This atlas determines transition functions:

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$$

such that

$$\psi_{\alpha\beta} := \varphi_\beta^{-1} \circ \varphi_\alpha : (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$$
$$\psi_{\alpha\beta}(p, v) = (p, g_{\alpha\beta}(p)(v))$$

If G acts on another space X on the left, then we can form the associated fiber bundle $\xi_X : X \to E' \to B$ from the data

$$\widetilde{\psi}_{\alpha\beta} : (U_\alpha \cap U_\beta) \times X \to (U_\alpha \cap U_\beta) \times X$$

$$\widetilde{\psi}_{\alpha\beta}(p,x) = (p, g_{\alpha\beta}(p)\cdot x)$$

In particular, ξ_G is a principal G-bundle, called the associated principal G-bundle.

On the other hand, suppose $\xi : G \to E \xrightarrow{\pi} B$ is a principal G-bundle, and suppose G acts on a space X on the left. We can define an action of G on $E \times X$ by:

$$g \cdot (p,x) = (pg^{-1}, gx)$$

Let $E \times_G X$ denote the quotient $(E \times X)/G$, and define $\pi_X : E \times_G X \to B$ by $\pi_X[p,x] = \pi(p)$. This makes a fiber bundle denoted by

$$\xi[X] : X \to E \times_G X \to B$$

This is called the associated fiber bundle.

It turns out these two constructions are inverses of each other, as stated in the following two theorems:

Theorem 2.1. *Let $\xi : F \to E \to B$ be a fiber bundle in which the structural group G acts freely and transitively on F, then*

$$\xi \cong \xi_G[F],$$

that is the associated fiber bundle of the associated principal G-bundle is equivalent to the original fiber bundle.

Theorem 2.2. *Let $\xi : G \to E \to B$ be a principal G-bundle and X a space on which G acts freely and transitively. Then*

$$\xi \cong \xi[X]_G,$$

that is the associated principal G-bundle of the associated fiber bundle is equivalent to the original principal G-bundle.

These theorems allow us to reduce the classification of (real, complex, quaternionic) vector bundles over a space to the classification of principal $(GL(n), U(n), Sp(n))$-bundles.

For a topological group G let us define a contravariant functor $k_G : \mathbf{hCW}_* \to \mathbf{Set}_*$ by

$$k_G(X) = \{\text{isomorphism classes of principal } G\text{-bundles over } X\}$$

For $X \xrightarrow{[f]} Y$ we define $k_G([f]) : k_G(Y) \to k_G(X)$ by

$$k_G([f])(\xi) = f^*\xi$$

This function is well-defined by the fact that pullbacks of bundles under homotopic maps are equivalent. The base point of $k_G(X)$ is the equivalence class of the trivial G-bundle $G \to G \times X \to X$.

Theorem 2.3. *The functor k_G satisfies the wedge axiom and the Mayer-Vietoris axiom.*

Therefore to each topological group G there is a CW-complex BG (determined up to homotopy type) and a principal G-bundle $G \to EG \to BG$ (this is the universal element) such that the natural transformation

$$T_G : [-, (BG, *)] \to k_G(-)$$

defined by $T_G(X)([f]) = \{f^*EG\}$ is an equivalence. Therefore principal G-bundles are classified by homotopy classes of maps into BG. For this reason we call BG the classifying space of G

and $EG \to BG$ the universal bundle of G. All principal G-bundles are pullbacks of the universal bundle. The classifying spaces $BO(n)$, $BU(n)$ and $BSp(n)$ are very important in K-theory, and we can easily find explicit constructions of these spaces (not done here). One form of the Bott periodicity theorem in K-theory can be stated as

$$\mathbb{Z} \times BU \simeq \Omega^2\, BU$$

$$\mathbb{Z} \times BO \simeq \Omega^4\, BSp$$

$$\mathbb{Z} \times BSp \simeq \Omega^4\, BO$$

Let us now see how $B-$ can be turned into a functor. For now we will think of $B-$ as a functor from **TopGrp**, the category of topological groups and continuous homomorphisms (these spaces are automatically based at the identity and homomorphisms preserve this point), to $\mathbf{hCW}_*$. Clearly $B-$ will take a topological group G to its classifying space BG. If $h : G \to G'$ is a morphism of topological groups, then we can define a natural transformation $T : k_G \to k_{G'}$ is the following way. Give a space X and a principal G-bundle ξ over X with transition functions $\{g_{\alpha\beta}\}$, let $T(X)(\xi)$ be the principal G'-bundle over X given by transition functions $\{h \circ g_{\alpha\beta}\}$. Then 1.4 gives us an induced map $(BG, *) \to (BG', *)$, which we call Bh.

Now let us apply this machinery to the case of G-bundles over the suspension, ΣX, of a space X. Here we are taking the reduced suspension $\Sigma X = X \wedge S^1$. By the adjoint relation (1.1) we see that principal G-bundles over ΣX are in one-to-one correspondence with

$$[(\Sigma X, *), (BG, *)] \cong [(X, *), (\Omega BG, *)]$$

We can write ΣX as the union of two contractible pieces that intersect in a space that deformation retracts onto X:

$$CX^+ = \{[x,t] \in \Sigma X : t \in (1/4, 1]\} \qquad CX^- = \{[x,t] \in \Sigma X : t \in [0, 3/4)\}$$

$$CX^+ \cap CX^- = \{[x,t] \in \Sigma X : t \in (1/4, 3/4)\} \simeq X$$

Therefore every G-bundle over ΣX can be constructed from a map $CX^+ \cap CX^- \to G$. For a map $f : X \to G$ let us define $\widetilde{f} : CX^+ \cap CX^- \to G$ by $\widetilde{f}[x,t] = f(x)$, and let $\xi(f)$ denote the G-bundle constructed via $\widetilde{f}$.

Lemma 2.1. *Two maps $f_0, f_1 : (X, x_0) \to (G, 1)$ are homotopic rel x_0 if and only if $\xi(f_0) = \xi(f_1)$.*

This lemma tells us that the natural transformation

$$T : [-, (G, 1)] \to k_G \circ \Sigma \tag{2.1}$$

is injective. It can also be shown that T is actually a natural equivalence, hence principal G-bundles over ΣX are classified by homotopy classes of maps of X into G.

Corollary 2.1. *G is homotopy equivalent to ΩBG.*

Proof. We have the following sequence of naturally equivalent functors

$$[-, \Omega BG] \longrightarrow [\Sigma -, BG] \longrightarrow k_G \circ \Sigma(-)$$

Since naturally equivalent functors have homotopy equivalent classifying spaces we see that ΩBG is the classifying object of $k_G \circ \Sigma$. The natural equivalence of (2.1) now shows that ΩBG is homotopy equivalent to G. □

So we can think of $B-$ as a one-sided inverse to Ω when applied to certain spaces. By the adjoint relation (1.1) we have

$$\pi_n(G) \cong \pi_n(\Omega BG) \cong \pi_{n+1}(BG)$$

In particular, if G is discrete, then

$$\pi_n(BG) \cong \pi_{n-1}(G) = \begin{cases} G & n = 1 \\ 0 & n > 1 \end{cases}$$

hence BG is a $K(G, 1)$ space.

Lemma 2.2. *For any topological group G, the total space EG of the universal bundle is weakly contractible.*

Proof. □

3 Examples

Here we will construct the universal bundles for some topological groups. Let us take the simplest non-trivial topological group, $G = S^1 = U(1)$. Then $BU(1)$ is a space such that $\pi_n(BU(1)) \cong \pi_{n-1}(G)$, so $BU(1)$ must be a $K(\mathbb{Z}, 2)$ space. This space must be homotopy equivalent to $\mathbb{C}P^\infty$, infinite complex projective space.