

## MAT 203 Final Exam Review Sheet

### 11: Vectors

- Given a vector  $v = \langle v_1, v_2, v_3 \rangle$  (also written  $v_1i + v_2j + v_3k$ ) it has magnitude  $\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$  and the unit vector in it's direction is  $\frac{v}{\|v\|}$ .
- Two vectors  $u$  and  $v$  are parallel if  $\frac{v}{\|v\|} = \pm \frac{u}{\|u\|}$
- The two vector products are the dot and cross product:

$$u \cdot v = u_1v_1 + u_2v_2 + u_3v_3$$

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- The angle  $\theta$  between two vectors  $u$  and  $v$  is given by  $\cos \theta = \frac{u \cdot v}{\|u\|\|v\|}$ , and  $u \cdot v = 0$  thus implies the two vectors are perpendicular. The projection of  $u$  onto  $v$  is given by  $\text{proj}_v u = \left(\frac{u \cdot v}{\|v\|^2}\right)v$ .
- The cross product is a vector perpendicular to both  $u$  and  $v$  and is the area of a parallelogram with  $u$  and  $v$  as sides. If you take  $|u \cdot (v \times w)|$  you get the volume of the parallelepiped having those three vectors as sides.
- The line in space through a point  $(x_0, y_0, z_0)$  and parallel to vector  $\langle a, b, c \rangle$  is given by parametric equations  $x(t) = x_0 + at$ ,  $y(t) = y_0 + bt$ , and  $z(t) = z_0 + ct$ .
- The plane which passes through  $(x_0, y_0, z_0)$  and has *normal* vector  $\langle a, b, c \rangle$  is given by the equation  $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$  (if you have two vectors in the plane, take the cross product to get a normal vector).
- The distance between a plane and a point  $Q$  is given by  $\|\text{proj}_n \overrightarrow{PQ}\|$ , where  $n$  is the normal vector to the plane and  $P$  is any point in the plane.
- The distance between a point  $Q$  and a line in space is given by  $\frac{\|\overrightarrow{PQ} \times u\|}{\|u\|}$ , where  $u$  is any vector parallel to the line and  $P$  is any point on the line.
- To sketch a surface in space, first sketch the traces (set each variable to 0 in turn and see what the graphs you get are - if you don't get anything, try setting it equal to a different constant instead). You should know the six types of elliptic curves in Figure 1.

| Graph                     | Equation  | Traces   |
|---------------------------|---|--|
| Ellipsoid                 | $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ | All Ellipses                                     |
| Hyperboloid of One Sheet  | $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ | Ellipse if $z$ is constant, hyperbola otherwise  |
| Hyperboloid of Two Sheets | $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ | Ellipse if $z$ is constant, hyperbola otherwise  |
| Elliptic Cone             | $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ | Ellipse if $z$ is constant, hyperbola otherwise  |
| Elliptic Paraboloid       | $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$                   | Ellipse if $z$ is constant, parabola otherwise   |
| Hyperbolic Paraboloid     | $z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$                   | Hyperbola if $z$ is constant, parabola otherwise |

Figure 1: The 6 types of Elliptic Curve

- The conversions from Rectangular to Cylindrical are:

$$r^2 = x^2 + y^2 \quad \tan(\theta) = \frac{y}{x} \quad z = z$$

- The conversions from Cylindrical to Rectangular are:

$$x = r \cos(\theta) \quad y = r \sin(\theta) \quad z = z$$

- The conversions from Rectangular to Spherical are:

$$\rho^2 = x^2 + y^2 + z^2 \quad \tan(\theta) = \frac{y}{x} \quad \cos(\phi) = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

- The conversions from Spherical to Rectangular are:

$$x = \rho \sin(\phi) \cos(\theta) \quad y = \rho \sin(\phi) \sin(\theta) \quad z = \rho \cos(\phi)$$

## 12: Vector-Valued Functions

- A vector valued function is a function  $\langle v_1(t), v_2(t), v_3(t) \rangle$ . The limit, derivative, and integral of a vector-valued function are each performed by doing the appropriate operation on each term individually and writing the result as a vector.
- The unit tangent vector to the curve  $r(t)$  is given by  $T(t) = \frac{r'(t)}{\|r'(t)\|}$ , while the principal unit normal vector is  $N(t) = \frac{T'(t)}{\|T'(t)\|}$ .
- The arc-length of a curve  $r(t)$  from  $t = a$  to  $t = b$  is given by

$$s = \int_a^b \|r'(t)\|$$

- The curvature of the curve  $r(t)$  (or in the last case, a curve  $y = f(x)$ ) can be computed by any of the following formulae:

$$K = \frac{\|T'(t)\|}{\|r'(t)\|}$$

$$K = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^2}$$

$$K = \frac{|y''|}{(1 + (y')^2)^{\frac{3}{2}}}$$

## 13: Functions of Several Variables

- To sketch a contour map of some function  $z = f(x, y)$ , just set  $z$  equal to different numbers (just pick some - I usually start with 0 and move out both directions - 1, -1, 2, -2). Solve for  $y$  and you'll have a 2-dimensional graph. Plot all those graphs, and you'll have the contour plot.
- If you want to take the limit of a function in two variables, you have to make sure it works along every curve through that point - if it doesn't agree along any two different curves, it doesn't exist. You can't just check the  $x$  and  $y$  directions. Since continuity uses limits, the same statement applies to it.
- You can take the partial derivative by treating all the other variables as constants and taking the derivative normally. If you're taking multiple partial derivatives, like  $\frac{\partial^2 f}{\partial x \partial y}$ , it doesn't matter which order you take them in if the function is continuous and the partial derivatives are continuous.
- The total differential of  $z = f(x, y)$  is

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

- We have several different chain rules for two variables. The cases are as follows:

–  $w = f(x(t), y(t))$ , then

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t}$$

–  $w = f(x(t, s), y(t, s))$ , then

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s}$$

– If we have the implicit function  $F(x, y) = 0$ , then

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

– If we have the implicit function  $F(x, y, z) = 0$ , then

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

- The gradient of  $f(x, y, z)$  is given by:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

and is the direction of maximum increase. It is also the normal vector for the level curve at that point.

- The directional derivative of  $f(x, y, z)$  in the direction of the vector  $u$  is given by  $D_u f = \nabla f \cdot u$ .
- To find the equation of the plane tangent to the surface given by  $F(x, y, z) = 0$ , just use a point on the surface and the gradient vector as your normal vector.
- To find extreme values on a region,
  1. First find the critical points (places where the first partials are 0).
  2. Then, taking the function restricted to the boundary (take the equation for the boundary of the region and plug it into  $f(x, y)$ ), find its critical points.
  3. Then add the corners to this list (places where any two pieces of the boundary intersect).
  4. Finally, plug each of these points back into the original equation. The one with the biggest value will be your absolute maximum, and the one with the smallest will be an absolute minimum.
- To determine whether the critical point  $(a, b)$  inside a region is a local maximum or local minimum (or saddle point), use the second partials test. Let  $D = \frac{\partial^2 f}{\partial x \partial x}(a, b) \frac{\partial^2 f}{\partial y \partial y}(a, b) - (\frac{\partial^2 f}{\partial x \partial y}(a, b))^2$ . Then
  - If  $D > 0$  and  $\frac{\partial^2 f}{\partial x \partial x} > 0$ , then  $f$  has a local minimum at  $(a, b)$ .
  - If  $D > 0$  and  $\frac{\partial^2 f}{\partial x \partial x} < 0$ , then  $f$  has a local maximum at  $(a, b)$ .
  - if  $D < 0$ , then  $f$  has a saddle point at  $(a, b)$ .
  - if  $D = 0$ , the test is inconclusive.
- To find the minimum or maximum of the optimization function  $f(x, y)$  subject to the constraint function  $g(x, y) = c$ , solve the system of equations:

$$\frac{\partial f}{\partial x}(x, y) = \lambda \frac{\partial g}{\partial x}(x, y)$$

$$\frac{\partial f}{\partial y}(x, y) = \lambda \frac{\partial g}{\partial y}(x, y)$$

$$g(x, y) = c$$

These are the potential solutions - plug them into the original function, and see which is largest/smallest, as before. The largest will be the maximum and the smallest the minimum, subject to the constraint.

## 14: Multiple Integration

- To perform iterated integration, you first need to find the bounds that you're integrating over. You'll need to write your integral like this:

$$\int_a^b \int_{c(x)}^{d(x)} \int_{e(x,y)}^{f(x,y)} g(x, y, z) dz dy dx$$

Note that the first set of limits has no variables, the second only has  $x$ , and the last has  $x$  and  $y$  - of course, these change if you change the order of integration - but you if you have variables after you've already integrated with respect to them, they'll end up in your answer.

- To find area or volume, integrate the function  $f(x, y, z) = 1$  over the region.
- If you change the function to polar coordinates, make sure you also change the limits of integration, and replace  $dx dy$  with  $r dr d\theta$ .
- The surface area  $S$  of the surface given by  $z = f(x, y)$  over the region  $R$  is given by

$$S = \iint_R \sqrt{1 + \frac{\partial f^2}{\partial x} + \frac{\partial f^2}{\partial y}} dA$$

- If you convert to cylindrical coordinates, replace  $dx dy dz$  with  $r dr d\theta dz$ .
- If you convert to spherical coordinates, replace  $dx dy dz$  with  $\rho^2 \sin \phi d\rho d\phi d\theta$ .
- If you do a generic change of variables  $T$ ,  $T(x) = g(u, v)$  and  $T(y) = h(u, v)$ , then

$$\iint_R f(x, y) dx dy = \iint_{T(R)} f(g(u, v), h(u, v)) \left| \det \begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{vmatrix} \right| du dv$$

## 15: Vector Analysis

- A vector field is something of the form  $\langle v_1(x, y, z), v_2(x, y, z), v_3(x, y, z) \rangle$ . You can draw one by plugging in points and drawing a small vector in the appropriate direction at that point (make vectors with larger magnitude longer).
- A vector field  $F = \langle M(x, y, z), N(x, y, z), P(x, y, z) \rangle$  is conservative if there is another function  $f$  such that  $F = \nabla f$ . In the plane, this is equivalent to  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ . In space, this is equivalent to  $\text{curl } F = 0$ .
- The curl of a vector field  $F(x, y, z) = \langle M(x, y, z), N(x, y, z), P(x, y, z) \rangle$  is given by

$$\text{curl } F = \det \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M(x, y, z) & N(x, y, z) & P(x, y, z) \end{vmatrix}$$

- The divergence of  $F = \langle M(x, y, z), N(x, y, z), P(x, y, z) \rangle$  is

$$\text{div } F = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

- If  $F$  is conservative, the line integral of  $F$  over any **closed** curve is 0 and the line integral over any curve depends only on the endpoints of the curve.
- Green's Theorem tells us that if  $C$  is a closed curve which runs counterclockwise around a region  $R$ , then

$$\int_C M dx + N dy = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

- Given a parametric surface with equations  $r = \langle x(u, v), y(u, v), z(u, v) \rangle$ , the normal vector is given by  $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$  and the surface area over a region  $R$  is given by

$$\iint_R \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| dA$$

- If  $f(x, y, z)$  is continuous over the surface  $S$  given by  $z = g(x, y)$  over the region  $R$ , the surface integral of  $f$  over  $S$  is given by

$$\iint_S f(x, y, z) dS = \iint_R f(x, y, g(x, y)) \sqrt{1 + \frac{\partial g^2}{\partial x} + \frac{\partial g^2}{\partial y}} dA$$

- The flux integral of a vector field  $F$  over a surface  $S$  given by  $z = g(x, y)$  (having normal vector  $N$ ) over the region  $R$  is given by

$$\iint_S F \cdot N dS = \iint_R F \cdot \left\langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \right\rangle$$

if the surface is oriented upwards, or by

$$\iint_S F \cdot N dS = \iint_R F \cdot \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, -1 \right\rangle$$

if it is oriented downwards

- The Divergence Theorem tells us that the flux integral of a vector field  $F$  over a closed surface  $S$  (oriented outward) around a region  $Q$  is given by

$$\iint_S F \cdot N dS = \iiint_Q \operatorname{div} F dV$$

- Stokes' Theorem tells us that the line integral of a vector field  $F$  over a closed curve  $C$  which runs counterclockwise around a surface  $S$  is the same as the flux integral of  $\operatorname{curl} F$  over  $S$ :

$$\int_C F \cdot dr = \iint_S \operatorname{curl} F \cdot N dS$$