MAT 132 Final Exam Review Sheet

Section 8.1

- A *sequence* is an ordered list of numbers. A *series* is the sum of an ordered list of numbers.
- Remember all of your rules of limits, but also remember when they don't apply. You only have $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$ if both limits exist. If either one doesn't exist, or is ∞, then you can't do this. The same goes for sums, differences, and quotients.
- Remember the Squeeze Theorem - if $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n = L$. Also remember the Monotonic Sequence Theorem - any bounded monotonic (which means always increasing or always decreasing) sequence is convergent.

Section 8.2

- A series $S_n = \sum_{i=0}^{\infty} a_i$ is convergent if $\lim_{n\to\infty} S_n$ exists, where $S_n = \sum_{i=0}^{n} a_i$. These are called the partial sums.
- The series $\sum_{n=0}^{\infty} ar^n$ is called the geometric series, and converges to $\frac{a}{1-r}$ if $r < 1$. If $r \geq 1$, it diverges.
- If a series $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Notice that this only tells you that a series is divergent if the limit of the sequence isn't 0. If the limit of the sequence is 0, it could still be divergent.
- The harmonic series $\sum_{n=0}^{\infty} \frac{1}{n}$ is divergent.

Section 8.3

- The Integral Test tells us that, if $f(x)$ is a function, then $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^{\infty} f(x)dx$ converges.
- The integral test, then, tells us that the p-series, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p < 1$.
- The Comparison Test lets us compare to series that we already know convergence or divergence of. It tells us that, if we have two sequences $a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ converges if $\sum_{n=1}^{\infty} b_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges if $\sum_{n=1}^{\infty} a_n$ diverges. That is, a series diverges if it's bigger than a divergent series, and a series converges if it's smaller than a convergent series.

Section 8.4

- The Alternating Series Test tells us that $\sum_{n=1}^{\infty} (-1)^n a_n$ converges if $a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$. So an alternating series converges if it's terms are getting smaller all the time, and if they eventually go to 0.
- We say a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.
- Any absolutely convergent series is convergent.
- The Ratio Test tells us that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$, and divergent if $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| > 1$. If the limit of the ratio is 1, it could be convergent or divergent.
- The Root Test tells us that if $a_n > 0$ for all n, then $\sum_{n=1}^{\infty} a_n$ is convergent if $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$. It is divergent if $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$. Like the Ratio Test, we know nothing if this limit is 1.

Section 8.5

- A power series is a series of the form $\sum_{n=1}^{\infty} c_n(x-a)^n$. We say that the power series is centered at a because it converges on an interval centered at a. The radius of that interval is called the radius of convergence, and the interval itself is called the interval of convergence.

- The best way to find the radius of convergence of a power series is to use the ratio test. The $\lim_{n\to\infty} |\frac{c_{n+1}x^{n+1}}{c_n x^n}|$ will be some function of x. Then find the values of x where that function is less than 1 - this will be the interval of convergence. At the endpoints of the interval, the function will be 1 - you need to check the actual series for convergence using some other test at the endpoints.

- If the power series converges for all x, we say the radius of convergence is ∞ and that the interval of convergence is $(-\infty, \infty)$.

Section 8.6

- If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$, then $f'(x) = \sum_{n=0}^{\infty} c_n n(x-a)^{n-1}$ and has the same radius of convergence as the power series for $f(x)$. That is, we can take the derivative of each term individually and we get the power series for the derivative.

- If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$, then $\int f(x) = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$ and has the same radius of convergence as the power series for $f(x)$. That is, we can take the antiderivative of each term individually and we get the power series for the antiderivative. As always when integrating, we add in the constant.

Section 8.7

- Taylor's theorem tells us that every function has a Taylor series centered at a - that is, that $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!}$ where $f^{(n)}(a)$ is the value of the nth derivative of f at a. We call this a Maclaurin series if $a = 0$.

- The four power series that you need to memorize are as follows. All of them have infinite radius of convergence except the geometric series (which has radius 1).
  - $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$
  - $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
  - $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
  - $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

Section 8.8

- The binomial series tells us that $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$, where $\binom{k}{n} = \frac{k(k-1)(k-2)\ldots(k-n+1)}{n!}$ and $\binom{k}{0} = 1$.

Section 8.9

- The k-th Taylor Polynomial centered at a, denoted $T_k(x)$, is just the k-th partial sum of the Taylor Series centered at a. That is, $T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)(x-a)^n}{n!}$.