MAT 132 Exam II Review Sheet

Things to Remember from Exam I

- Substitution: $\int_a^b f(u(x))dx = \int_{u(a)}^{u(b)} \frac{f(u)}{u'(x)} du$
- Integration by Parts: $\int_a^b u(x)v'(x)dx = u(x)v(x)|_a^b - \int_a^b v(x)u'(x)dx$

Section 7.1

- A differential equation is an equation which defines some derivative of y in terms of x, y, y', y'', etc.
- The actual solution of a differential equation also depends on some initial value, since the functions could be very different depending on where they start, even though they have the same derivatives.
- To verify a solution of a differential equation, show that the function y(x) given satisfies the differential equation given, and show that the initial value is the same as the one given. For example:
  - Problem: Show that the function $y(t) = e^{rt} + \sin rt$ satisfies the differential equation $y'''' = r^4 y$ with initial value $y(0) = 1$.
  - Solution: First we compute the 4th derivative:
    * $y'(t) = re^{rt} + r\cos rt$
    * $y''(t) = r^2 e^{rt} - r^2 \sin rt$
    * $y'''(t) = r^3 e^{rt} - r^3 \cos rt$
    * $y''''(t) = r^4 e^{rt} + r^4 \sin rt$

    Then we note that $y''''(t) = r^4(e^{rt} + \sin rt) = r^4 y$. So it satisfies the equation. Now we plug in $t = 0$ and compute $y(0) = e^{r0} + \sin 0 = 1$ and we're done.

Section 7.2

- Direction Fields: To draw a direction field, take each point (x, y) and plug it into the given differential equation. This will give you some value for y'. So this will be the slope of the tangent line at that point. Draw a short line with that slope centred on the point (x, y).

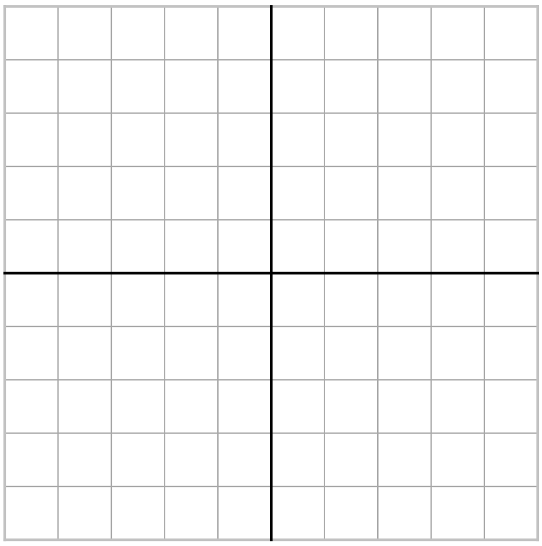

Figure 1: Draw in the direction field for $y' = \frac{xy}{4}$

- Euler's Method: We start off with an initial condition (x_0, y_0), an equation for $y'(x, y)$, a step size Δx, and an x that we want the value of y at. Then we find new values (x_1, y_1) by the formulae:
  - $x_1 = x_0 + \Delta x$

- $y_1 = y_0 + \Delta x(y'(x_0, y_0))$

and repeat until x_n is the value of x that we want $y(x)$ for. For Example, if $(x_0, y_0) = (1, 1)$, $y'(x, y) = \frac{xy}{2}$, and $\Delta x = 1$, approximate $y(5)$:

- $x_1 = 1 + 1 = 2$; $y_1 = 1 + 1(\frac{1*1}{4}) = \frac{5}{4} = 1.25$
- $x_2 = 2 + 1 = 3$; $y_2 = \frac{5}{4} + 1(\frac{2*1.25}{4}) = \frac{15}{8} = 1.875$
- $x_3 = 3 + 1 = 4$; $y_3 = \frac{15}{8} + 1(\frac{3*\frac{15}{8}}{4}) = \frac{105}{32} = 3.28125$
- $x_4 = 4 + 1 = 5$; $y_4 = \frac{105}{32} + 1(\frac{4*\frac{105}{32}}{4}) = \frac{105}{16} = 6.5625$

Section 7.3

- Separable Equations: To solve a differential equation of the form $\frac{dy}{dx} = g(x)f(y)$, we rewrite it in the form $\frac{1}{f(y)}dy = g(x)dx$ and then integrate both sides of the equation. Be sure to remember your $+C$ when you integrate - that C is how we represent the initial value!
  - Problem: Find a solution to the equation $y' = x^2y + 2y$ with initial value $y(0) = 5$.
  - Solution: First, note that y' is another way of writing $\frac{dy}{dx}$. Then we have $\frac{dy}{dx} = y(x^2 + 2)$, we divide both sides by y to get $\frac{dy}{ydx} = x^2 + 2$. We multiply both sides by dx and we get $\frac{dy}{y} = x^2 + 2dx$. Now we integrate both sides and get $\ln|y| = \frac{x^3}{3} + 2x + C$. Now, to solve for y, we exponentiate both sides and get $|y| = e^{\frac{x^3}{3}+2x+C}$. Equivalently, we get $|y| = e^{\frac{x^3}{3}+2x}e^C$, and since e^C is just a different constant, we can rewrite this as $y = Ce^{\frac{x^3}{3}+2x}$ (the absolute value sign goes away since we can make this C a positive or negative number). Now we need to figure out what that C is. We do this by plugging in $x = 0, y = 5$ and solving for C. This gives us $5 = Ce^{\frac{0^3}{3}+2*0} = C$. So our solution is $y = 5e^{\frac{x^3}{3}+2x}$. (Take the derivative and verify this is a solution to the equation!)

Section 7.4

- When we say something "grows exponentially" or "grows at a rate proportional to its size," we mean the same thing. That is, we mean that $P'(t) = kP(t)$. We call k the relative growth rate or the constant of proportion. The solution of this equation, which we get by separating and integrating, is $P(t) = P_0e^{kt}$, where P_0 is the initial value. Things that grow exponentially include bacterial populations, radioactive materials (which decay exponentially - meaning that k is negative), money compounded continuously, and zombies.
  - Assume a zombie infects two new zombies every 15 minutes. If we start with 200 of the ravening flesh-devouring monsters, give an equation which tells us how many we have after t hours. After how many hours will we have enough zombies to fill Yankee Stadium (seats 57,545)?
  - Solution: The model, as stated above, is $P' = kP$, and the solution is $P = P_0e^{kt}$. P_0 is given to be 200. We need to determine k, using the data point we're given. This tells us that after 15 minutes, the number of zombies triples (each original zombie, and the two new shambling corpses he/she/it has created). So $P(\frac{1}{4})$ (remember to work in hours!) is 600. Therefore, $600 = 200e^{k\frac{1}{4}}$. We solve for k by dividing both sides by 200 to get $3 = e^{\frac{k}{4}}$, so $\ln 3 = \frac{k}{4}$, which tells us that $k = 4\ln 3 \approx 4.3944$. So we have the equation $P(t) = 200e^{4\ln 3t}$. To find out when the zombies have a standing room only crowd, we set this equation equal to $57,545$ and solve for t. So let $57545 = 200e^{4\ln 3t}$. Then $t = \frac{\ln\frac{57545}{200}}{4\ln 3} \approx 1.288$. So just over an hour and 15 minutes later.
- Don't forget these other formulae:
  - Newton's Law of Cooling: $\frac{du}{dt} = k(u - A)$ where u is the temperature of the object and A is the ambient temperature. $u(0)$ is thus the initial temperature of the object.
  - Compounded Interest: $\frac{dA}{dt} = rt$, where A is the amount of money you have and r is the rate of interest. $A(0)$ is thus your principal (the amount of money you start with).

Appendix H

- When we talk about a point in polar coordinates, we think of it as (r, θ), where r is the radius (how far it is from the origin) and θ is the angle (that is, if you drew a straight line from the origin to the point, what angle would it make with the x-axis). If r is negative, we go the opposite direction that we would expect from the angle (think of it as being the same angle, but taken in the *negative* direction).

- To convert from polar (r, θ) to cartesian (x, y), we use the formulae:
  - $x = r\cos\theta$; $y = r\sin\theta$
  - $r = \sqrt{x^2 + y^2}$; $\theta = \arctan\frac{y}{x}$
- To graph a polar curve, first graph it on cartesian coordinates (with (r, θ) just corresponding to (x, y). Then think about each section and what it would look like (for example - starts at 0, goes up to 1, then back to 0 would give you a loop going out to a radius of 1 and then coming back into the origin)

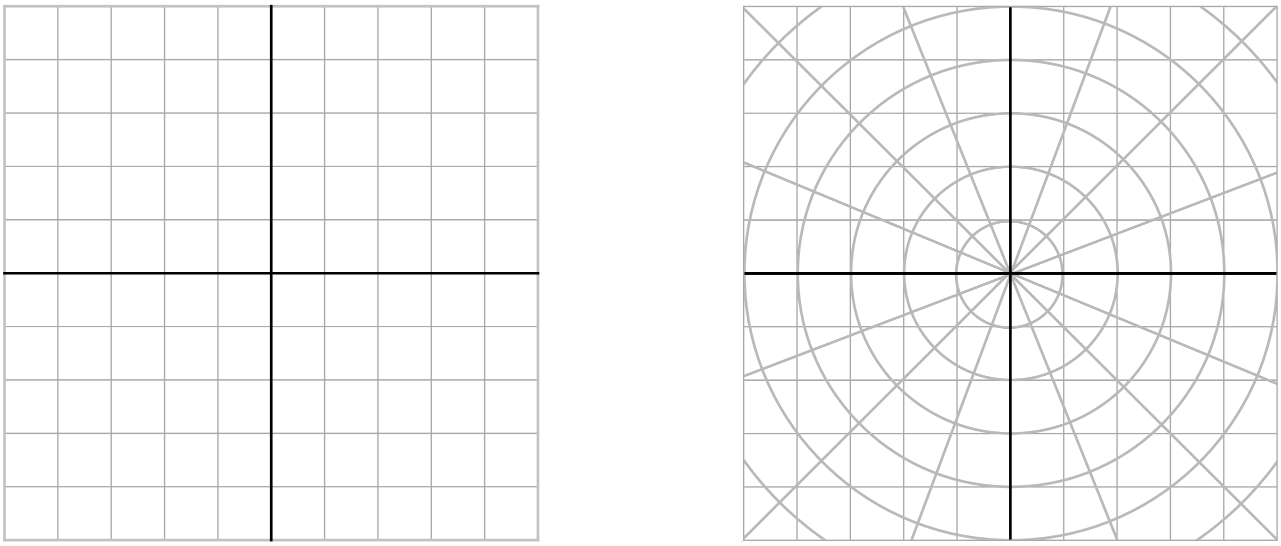

Figure 2: Draw in a graph of $r = 3\cos 3\theta$ (first in cartesian coordinates, then in polar)

- The area inside a polar curve is given by the formula $A = \int_a^b \frac{1}{2}r^2(\theta)d\theta$. Here a and b are the angles where the area starts and ends.
  - Compute the area swept out by the curve $r = \cos 2\theta$ between $\frac{-\pi}{4}$ and $\frac{\pi}{4}$.
  - Solution: The area will be $\frac{1}{2}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \cos^2(2\theta)d\theta$. Recall the half-angle identity: $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$. Then we have that this is $\frac{1}{4}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} 1+\cos(4\theta)d\theta$. So the answer will be $\frac{1}{4}((\frac{\pi}{4}+\frac{1}{4}\sin(\pi))-(\frac{-\pi}{4}+\frac{1}{4}\sin(-\pi))) = \frac{1}{4}(\frac{2\pi}{4}) = \frac{\pi}{8}$.

Appendix I

- A complex number is a number of the form $a + bi$, where a and b are real numbers and i is the $\sqrt{-1}$.
- The conjugate of a complex number $z = a + bi$, denoted by $\overline{z}$ is $a - bi$.
- The magnitude of a complex number $z = a + bi$, denoted by $|z|$, is $\sqrt{a^2 + b^2}$. It can also be written $|z| = z\overline{z}$
- We can write z in polar form as $z = r\cos\theta + ir\sin\theta$. Then we discover De Moivre's Theorem: $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$.
- A consequence of De Moivre's theorem is that all of the nth roots of a complex number z are evenly spaced in the complex plane, with the real nth root of $|z|$ being the radius and the rest being spaced $\frac{2\pi}{n}$ apart from each other.
- Some more complicated math we haven't quite gotten to yet gives us the fact that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.