MAT 125 Final Exam Review Sheet

3.5: The Chain Rule

- The derivative of $f(g(x))$ is $f'(g(x))g'(x)$.
- The derivative of a^x is $a^x \ln(a)$.

3.6: Implicit Differentiation

- If we have an equation in terms of x and y, we can find the derivative of y with respect to x by performing implicit differentiation. To do this, we think of y as a function of x - that is, $y(x)$. We then differentiate each term of the equation individually and solve for $y'(x)$.
- If we have a term containing both y and x, we need to use either the chain rule or the product rule (possibly both). For example, xy would need to use the product rule, since it is the function x times the function $y(x)$. y^2 would require the chain rule, since it is the composition of x^2 and $y(x)$.
- The derivative of $\sin^{-1}(x)$ is $\frac{1}{\sqrt{1-x^2}}$. The derivative of $\cos^{-1}(x)$ is $\frac{-1}{\sqrt{1-x^2}}$. The derivative of $\tan^{-1}(x)$ is $\frac{1}{1+x^2}$.

3.7: Derivatives of Logarithmic Functions

- The derivative of $\log_a(x)$ is $\frac{1}{x\ln(a)}$. Therefore, the derivative of $\ln(x)$ is $\frac{1}{x}$.
- The derivative of $\ln(f(x))$ is $\frac{f'(x)}{f(x)}$.
- To do logarithmic differentiation (usually necessary if you have something of the form $f(x)^{g(x)}$), take the logarithm of both sides, simplify using logarithm rules, and then perform implicit differentiation.

4.1: Related Rates

- The strategy for related rates problems:
  - Draw the picture at the time when you want to figure out the rate. For example, if it says two hours after things start moving, draw them at their positions after two hours.
  - Label everything. Include rates with little arrows showing what direction things are going (this will help you decide whether it should be negative - if the length is getting shorter - or positive - if the length is getting longer). If something changes with time, give it a function name like $a(t)$.
  - Decide what you need. Usually this will be the rate at which one of your sides is changing length. This will be the derivative of the function that you assigned to the length of that side. Be very careful here. Take the time to make sure you understand what's changing and what's constant; mistakenly treating something that's changing as a constant will give you the wrong answer.
  - Find a formula that relates what you have to what you need. Fill in values for things that aren't changing, and functions for things that are changing.
  - Using implicit differentiation, solve for the derivative that you want.
  - Fill in the values to get a result.

4.2: Maximum and Minimum Values

- A function f has an absolute maximum at c if $f(c) \geq f(x)$ for all x in the domain. f has an absolute minimum at c if $f(c) \leq f(x)$ for all x in the domain.
- A function f has a local maximum at c if $f(c) \geq f(x)$ for all x close to c. f has a local minimum at c if $f(c) \leq f(x)$ for all x close to c.
- A critical number of f is a value x such that $f'(x) = 0$ or $f'(x)$ does not exist.
- If f has a local maximum or minimum at c, then c is a critical number.

- To find the global maximum of f on $[a, b]$, find the value of f at all critical numbers and at the endpoints a and b. The global maximum will be at one of these points - the one with the largest value. The global minimum will be at the one with the lowest value.

4.3: Derivatives and Shapes of Curves

- The Mean Value Theorem tells us that if we have $f(x)$ differentiable on $[a, b]$, then there is some point c between a and b such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
- The relationship between a function and its graph is summarized in the table below.

f	f'	f"
Increasing	Positive	
Decreasing	Negative	
Maximum	0 and Decreasing	Negative
Minimum	0 and Increasing	Positive
Concave Up	Increasing	Positive
Concave Down	Decreasing	Negative
Inflection Point	Horizontal	0

4.5: Indeterminate Forms and L'Hospital's Rule

- Suppose that we have $\lim_{x \to a} \frac{f(x)}{g(x)}$ where either $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ or $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$. Then we can apply L'Hospital's Rule: $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ if the limit exists.
- If have a limit of a product $\lim_{x \to a} f(x)g(x)$ where $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$ we can rewrite it as $\lim_{x \to a} \frac{g(x)}{\frac{1}{f(x)}}$ and apply L'Hospital's Rule.
- If we have a limit of a difference $\lim_{x \to a} f(x) - g(x)$ where $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$, then we can try writing them as fractions, putting them over a common denominator, and using L'Hospital's rule.
- If we have a limit of a power, then we can use logarithms to reduce it to one of the previous two cases.

4.6: Optimization Problems

- The steps for solving optimization problems are similar to those from 4.1:
  - Draw the picture.
  - Determine what you're optimizing the value of. This is what you want to get as largest or smallest. Give this value a function name
  - Give everything else that changes a variable name (say y and x).
  - You should have two functions from the problem: a constraint function and an optimization function. Determine what these functions are based on what you have and what you want to get.
  - You want to solve the constraint function for one of the two variables (we'll solve for y) and then plug in what you get into the optimization function.
  - Now you have your optimization function in one variable. Take the derivative.
  - Find the absolute minimum or maximum (depending on whether you want it largest or smallest) of your optimization function as we did in section 4.2. If you don't have an interval $[a, b]$, this will be the local maximum or minimum - so you'll need to solve $f'(x) = 0$ for x.

4.8: Newton's Method

- Newton's method is a way of finding the root of a function - that is, where $f(x) = 0$. The steps are as follows:
  - Choose a value - we'll call it x_0 - close to where you think the root is - you can look at the graph, or just guess. The closer you start, the less time it will take to find a good approximation.

- Now we define x_n as $x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$. So $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, etc.

- As n gets larger, the value of x_n will get closer and closer to the actual location of the root. We can find the correct value to m decimal places by repeating the process until x_n and x_{n+1} agree on the first m decimal places. Then x_n is your approximation.