MAT 123 Final Exam Review Sheet

1.1: Graphs

- An ordered pair is a pair (x, y) on a graph.
- An x-intercept is a point at which the graph intercepts the x-axis $(y = 0)$.
- A y-intercept is a point at which the graph intercepts the y-axis $(x = 0)$.

1.2: Lines and Slope

- The slope of the line between (x_1, y_1) and (x_2, y_2) is $\frac{y_2-y_1}{x_2-x_1}$.
- A horizontal line has slope 0.
- a vertical line has undefined slope.
- A line with slope m passing through (x_1, y_1) has equation $y - y_1 = m(x - x_1)$.
- A line with slope m and y-intercept b has equation $y = mx + b$.
- Parallel lines have the same slope.
- Perpendicular lines have slopes that are negative reciprocals of each other (that is, if one has slope m, the other has slope $\frac{-1}{m}$).

1.3: Distance & Midpoint Formulas; Circles

- The distance between the points (x_1, y_1) and (x_2, y_2) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- The midpoint of a line segment between (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.
- A circle with center (h, k) and radius r has equation $(x - h)^2 + (y - k)^2 = r^2$.
- If you're given an equation of a circle in the form $x^2 + y^2 + ax + by + c = 0$, we can find the standard form by completing the square for both x and y. This will give you a circle with center $(\frac{a}{2}, \frac{b}{2})$ and radius $\sqrt{-c + (\frac{a}{2})^2 + (\frac{b}{2})^2}$.

1.4: Basics of Functions

- A function is a map from one set (called the domain) to a second set (called the range) where each element in the domain is mapped to exactly one element in the range.
- To determine if something is a function, see if the same value of x is ever mapped to two or more different values of y. If it is, then the equation is not a function.
- The difference quotient is defined as $\frac{f(x+h)-f(x)}{h}$.
- A piecewise function is defined as one function on part of its domain and other functions on other parts of its domain.
- The domain of a function is the set of points x where $f(x)$ is defined. For this class, you need to know that $\sqrt{(x)}$ is defined if $x \geq 0$, that $log(x)$ is defined for $x > 0$, and that a function is undefined if its denominator is 0.

1.5: Graphs of Functions

- Another way to check if a graph is a function is to see if a vertical line ever passes through more than one point of the graph.
- A function is increasing on an interval if, for all x_1, x_2 in the interval, if $x_1 < x_2$, then $f(x_1) < f(x_2)$. A function is decreasing if $x_1 < x_2$, then $f(x_1) > f(x_2)$. A function is constant if $f(x_1) = f(x_2)$.

- A relative minimum is a point at which the function stops decreasing and starts increasing. A relative maximum is a point at which the function stops increasing and starts decreasing.
- The average rate of change between x_1 and x_2 is the slope of a line between $(x_1, f(x_1))$ and $(x_2, f(x_2))$.
- A function is called even if $f(-x) = f(x)$. Even functions are symmetric with respect to the y-axis. A function is called odd if $f(-x) = -f(x)$. Odd functions are symmetric with respect to the origin.
- A step function is a function that is defined as a piecewise function, where all pieces are constant functions.

1.6: Transformations of Functions

- You need to know the graphs of common functions given on P. 184-5.
- Let $f(x)$ be a function. Then, if a is positive, $f(x) + a$ has the same graph, but shifted up by a units. If a is negative, $f(x) + a$ has the same graph, but shifted down by a units.
- Let $f(x)$ be a function. Then, if c is positive, $f(x + c)$ has the same graph, but shifted *left* by c units. If c is negative, $f(x + c)$ has the same graph, but shifted *right* by c units.
- The graph of $f(-x)$ is the graph of $f(x)$ reflected about the y-axis.
- The graph of $-f(x)$ is the graph of $f(x)$ reflected about the x-axis.
- If $c > 0$, the graph of $cf(x)$ is either a stretching or shrinking of the graph of $f(x)$. If $c > 1$, then the graph is stretched vertically by c. If $c < 1$, the graph is shrunk vertically by c.

1.7: Combinations and Compositions of Functions

- $f(x) + g(x) = (f + g)(x)$
- $f(x) - g(x) = (f - g)(x)$
- $f(x)g(x) = fg(x)$
- $\frac{f(x)}{g(x)} = \frac{f}{g}(x)$
- To form the composite "f of g", written $f \circ g(x)$, we take the function $g(x)$ and write it anywhere x appears in $f(x)$. For example, if $f(x) = x^2$ and $g(x) = \sin(x)$, then $f \circ g(x) = (\sin(x))^2$. The domain consists of all points x where x is in the domain of $g(x)$ and $g(x)$ is in the domain of $f(x)$.

1.8: Inverses

- Finding Inverses: To find the inverse of $y = f(x)$, exchange the y and the x and solve for y. If you can't get y alone on one side of the equation, then $f(x)$ does not have an inverse.
- To verify that the inverse of $f(x)$ is $g(x)$, take $f \circ g(x)$. If they are inverses, this will be equal to x.
- Another quick way to check for inverses is the horizontal line test. If you can't draw a horizontal line that goes through two points of the graph, then $f(x)$ is invertible. If you can, it's not.
- The graph of $f^{-1}(x)$ (this is how we write the inverse of $f(x)$) is the reflection of the graph of $f(x)$ over the line $y = x$.

2.2: Quadratic Functions

- The standard form of a quadratic function is $a(x - h)^2 + k, a \neq 0$. This parabola will have vertex (h, k) and line of symmetry $x = h$. If $a > 0$, then it opens upward; if not, it opens downward.
- If you write the quadratic function as $f(x) = ax^2 + bx + c$, then the vertex is $(-\frac{b}{2a}, f(-\frac{b}{2a}))$.
- In either case, if $a > 0$, the vertex is a minimum; if $a < 0$, the vertex is a maximum.

2.3: Polynomial Functions and their Graphs

- Steps for graphing a polynomial function:
  1. Use the leading coefficient test to determine the end behaviour. If your largest exponent is n and the coefficient of the x^n term is a_n, then the end behaviour is:
     - n **even,** $a_n > 0$: rises to the left and right
     - n **odd,** $a_n > 0$: falls to the left and rises to the right
     - n **even,** $a_n < 0$: falls to the left and right
     - n **odd,** $a_n < 0$: rises to the left and falls to the right
  2. Find the x-intercepts by setting $f(x) = 0$ and solving for x. When you factor the polynomial, the exponent of the factor $(x - a)$ is called the multiplicity of the root a. If the multiplicity of a root is even, the graph touches the x-axis and turns around. If it is odd, then it crosses the x-axis. If it is greater than 1, then the graph flattens out at the x-axis (but still crosses or turns around, depending on whether it is odd or even).
  3. Find the y-intercept by computing $f(0)$.
  4. Check to see if the graph has y-axis symmetry (if $f(-x) = f(x)$).
  5. Check to see if the graph has origin symmetry (if $f(-x) = -f(x)$).
  6. Note that the graph will turn no more than $n-1$ times, where n is the largest exponent. (It may turn less times, though!)

2.6: Rational Functions and their Graphs

- If $f(x) = \frac{p(x)}{q(x)}$ is a rational function, then x is in the domain of f if it is in the domain of p and q, and if $q(x) \neq 0$.
- An *asymptote* is a line that the graph comes infinitely close to but never reaches or crosses.
- In order to graph the rational function $f(x) = \frac{p(x)}{q(x)}$:
  1. Check for symmetry. If $f(-x) = f(x)$, then the graph is symmetric across the y-axis. If $f(-x) = -f(x)$, then the graph is symmetric about the origin.
  2. Find the y-intercept (if there is one) by evaluating $f(0)$.
  3. Find the x-intercepts (if there are any) by solving $p(x) = 0$ for x.
  4. Find any vertical asymptotes by solving $q(x) = 0$ for x.
  5. Determine whether there is a horizontal or slant asymptote by examining the leading terms of p and q. If the leading term of p is $p_n x^n$ and the leading term of q is $q_m x^m$, then:
     - If $n < m$, the x-axis is a horizontal asymptote.
     - If $n = m$, the line $y = \frac{p_n}{q_n}$ is a horizontal asymptote.
     - If $n > m$, the graph has no horizontal asymptote. However, if $n = m+1$, then there is a slant asymptote. You find the slant asymptote by dividing p by q and ignoring the remainder.
  6. Plot at least one point between and beyond each x-intercept and vertical asymptote.

3.1 Exponential Functions

- An exponential function is a function $f(x) = a^x$, where $a > 0$ and $a \neq 1$.
- If $1 < a < b$, then $a^x < b^x$ for all $x > 0$ and $a^x > b^x$ for all $x < 0$.
- If $1 > a > b$, then $a^x < b^x$ for all $x > 0$ and $a^x > b^x$ for all $x < 0$.
- $y = 0$ is a horizontal asymptote for a^x (note, though, that if you have $a^x - b$, then $y = -b$ is the horizontal asymptote).

3.2 Logarithmic Functions

- For $x > 0$ and $b > 0$, $b \neq 1$, we define the logarithm of x with base b as $y = \log_b x$. This is the same as $b^y = x$.

- $\log_b x$ is the inverse of b^x.
- $\log_b b = 1$.
- $\log_b 1 = 0$.
- $\log_b b^x = x$.
- $b^{log_b x} = x$.
- The graph of $y = \log_b x$ has x-intercept 1 and a vertical asymptote at $y = 0$. If $b > 1$, the function is increasing. If $b < 1$, the function is decreasing.
- The common log, which has base 10, is written $\log x$ (so if there is no b, assume it is 10).
- The natural log, which has base $e \approx 2.71828$ is written $\ln x$.

3.3: Properties of Logarithms

- $\log_b MN = \log_b M + \log_b N$
- $\log_b \frac{M}{N} = \log_b M - \log_b N$
- $\log_b M^p = p \log_b M$
- $\log_b M = \frac{\log_a M}{\log_a b}$

3.4: Exponential and Logarithmic Equations

- To solve an exponential equation, get the exponential part on one side of the equation (and combine all exponential terms into one using rules of exponents) and then take the natural logarithm (ln) of each side. This will get rid of the exponent and you can solve it normally.
- To solve a logarithmic equation, get the logarithmic part on one side of the equation (and combine all logarithmic terms into one using rules of logarithms). Then exponentiate each side by whatever number is the base of your logarithm. This will get rid of the logarithm, and you can solve normally.

3.5: Modeling with Exponential/Logarithmic Equations

- If something grows or decays exponentially, then it is modeled by $A = A_0 e^{kt}$, where A_0 is the starting amount and A is the amount obtained/left at time t. k is a constant called the growth/decay constant. If k is positive, this function models exponential growth, and if k is negative, it models exponential decay.
- For a half-life problem, you'll be given that half of the initial amount decays in t_0 years. Then you should solve the equation $\frac{A_0}{2} = A_0 e^{kt_0}$, or, equivalently $\frac{1}{2} = e^{kt_0}$ for k to get the decay function.
- The logistic growth function $A = \frac{c}{1+ae^{-bt}}$ grows exponentially but then slows down.
- Newton's Law of cooling tells us that $T = C + (T_0 - C)e^{kt}$, where T_0 is the initial temperature, C is the ambient temperature, and k is some constant (which will be less than 0) which is determined by the material cooling. In cooling problems, you'll first solve for k and then use it to determine temperatures after a certain amount of time.

4.1: Angles

- An angle θ is in standard position if one ray is on the positive x-axis and the angle other is θ degrees counterclockwise from it.
- Two angles are coterminal if their difference is some multiple of $360°$ or 2π radians.
- The complement of θ is $90° - \theta$. The supplement of θ is $180° - \theta$.
- A radian is the angle which sweeps out an arc equal to the radius of a circle. If θ is an angle in radians, then the length of any arc s is equal to $r\theta$, where r is the radius of the circle.

- If θ is in radians, then it is equal to $\frac{180}{\pi}\theta$ degrees. If θ is in degrees, then it is equal to $\frac{\pi}{180}\theta$ radians.
- If a circular object is rotating at ω radians per unit time, then a point on the edge has velocity v where $v = r\omega$ and r is the radius of the object.

4.2: Trigonometric Functions on the circle

- If (x, y) is a point on the unit circle, and the ray from the origin to (x, y) forms angle θ with the positive x-axis, then
  - $\sin\theta = y$
  - $\cos\theta = x$
- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\csc\theta = \frac{1}{\sin\theta}$
- $\sec\theta = \frac{1}{\cos\theta}$
- $\cot\theta = \frac{1}{\tan\theta}$
- The domain of sine and cosine is all real numbers. The range is $[-1, 1]$.
- The cosine and secant functions are even. The sine, cosecant, tangent, and cotangent are odd.
- The Pythagorean identities (which we get by dividing the Pythagorean theorem $x^2 + y^2 = r^2$ by r^2, x^2, and y^2) tell us that:
  - $\sin^2 t + \cos^2 t = 1$
  - $1 + \tan^2 t = \sec^2 t$
  - $\cot^2 t + 1 = \csc^2 t$
- The sine, cosine, cosecant, and secant have period 2π. That is, $f(x) = f(x+2\pi)$ for all x. The tangent and cotangent have period π.

4.3: Right Triangle Trigonometry

- If we have a right triangle with sides a,b, and c, where the hypoteneuse has length c and θ is the angle formed by the hypoteneuse and the side with length b, then:
  - $\sin\theta = \frac{a}{c}$
  - $\cos\theta = \frac{b}{c}$
  - $\tan\theta = \frac{a}{b}$
- $30° = \frac{\pi}{6} \sin\frac{\pi}{6} = \frac{1}{2}, \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$.
- $45° = \frac{\pi}{4} \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}, \tan\frac{\pi}{4} = 1$.
- $60° = \frac{\pi}{3} \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \cos\frac{\pi}{3} = \frac{1}{2}, \tan\frac{\pi}{3} = \sqrt{3}$.

4.4: Trigonometric Functions of any angle

- Given an angle in standard position, the reference angle is the angle that its second ray forms with the x-axis.
- The value of a trigonometric function at a given angle will either be the same as the value at the reference angle, up to a sign change.
- In Quadrant I, all trig functions are positive. In Quadrant II, just sine and cosecant are positive. In Quadrant III, just tangent and cotangent are positive. In Quadrant IV, just cosine and secant are positive.

4.5: Graphs of Sine and Cosine

- Study the graphs given in section 4.5.
- The amplitude is the maximum value achieved by a sine or cosine function. If $f(x) = A \sin Bx$, the amplitude is A.
- The wavelength is the distance between wave crests on the graph. If $f(x) = A \sin Bx$, the wavelength will be $\frac{2\pi}{B}$.
- If $f(x) = A \sin(Bx + C)$, then the graph looks like the graph of sin but with amplitude A, wavelength $\frac{2\pi}{B}$, and shifted over by $\frac{C}{B}$. To the left if $\frac{C}{B}$ is positive, and to the right if $\frac{C}{B}$ is negative.
- All of these same facts are true for cos.
- As always, adding a constant $f(x) + D$ is a vertical shift.

4.6: Graphs of Other Trig Functions

- Examine the graphs given in section 4.6.
- To graph tangent, secant, cosecant, and cotangent, one must first determine where the vertical asymptotes are. For $f(x) = \tan(Bx - C)$ or $f(x) = \sec(Bx - C)$, you do this by solving $Bx - C = \frac{\pi}{2}$ and $Bx - C = -\frac{\pi}{2}$. For cotangent and cosecant, you do this for π rather than $\frac{\pi}{2}$.
- Remember that for cosecant, secant, and cotangent, you're graphing the reciprocal of the sine, cosine, or tangent respectively. Therefore, if sine is y at a certain x, then cosecant will be $\frac{1}{y}$ (undefined if $y = 0$. The same is true for all of these functions. Use these relationships to help you graph.

4.7: Inverse Trigonometric Functions

- The inverse functions take x to θ, where, for example, $\sin(\theta) = x$
- Since the trigonometric functions fail the horizontal line test, we restrict the range of the inverse trig functions. $\sin^{-1}$ and $\tan^{-1}$ have range $[\frac{-\pi}{2}, \frac{\pi}{2}]$. $\cos^{-1}$ has range $[0, \pi]$.