Computing Symplectic Cohomology of Affine Varieties

(using Spectral Sequences)

http://www.math.stonybrook.edu/~markmclean/talks/spectralsequence-shorterversion.pdf

Related Projects (in progress)

- Diogo-Lisi
- Ganatra, Pomerleano
- Sheridan, Borman
- Hülya Argüz
- Joint work with Tehrani, Zinger.

- *“The words ‘spectral sequence’ strike fear into the hearts of many hardened mathematicians. These notes will attempt to demonstrate that spectral sequences are not so scary, and also very powerful.”* - M. Hutchings

- *"The words 'spectral sequence' strike fear into the hearts of many hardened mathematicians. These notes will attempt to demonstrate that spectral sequences are not so scary, and also very powerful."* - M. Hutchings
- *"A spectral sequence is an algebraic object, like an exact sequence, but more complicated"* - J. F. Adams.

- *"The words 'spectral sequence' strike fear into the hearts of many hardened mathematicians. These notes will attempt to demonstrate that spectral sequences are not so scary, and also very powerful."* - M. Hutchings
- *"A spectral sequence is an algebraic object, like an exact sequence, but more complicated"* - J. F. Adams.
- *"After my article was published, John Harper sent me email and said that when he was a graduate student back in the 1960s, he personally asked Leray about the term 'spectral' and in particular asked whether it had something to do with the spectrum of an operator. Leray began his reply by saying, "Non"; unfortunately, before he could continue, some professors approached and interrupted the conversation."* -Source: Timothy Chow/ Mathoverflow.net

A *spectral sequence* is a sequence of bigraded chain complexes. This is page 0.

$$\begin{array}{ccccccc}
E_0^{-1,2} & E_0^{0,2} & E_0^{1,2} & E_0^{2,2} & E_0^{3,2} & E_0^{4,2} & E_0^{5,2} \\
E_0^{-1,1} & E_0^{0,1} & E_0^{1,1} & E_0^{2,1} & E_0^{3,1} & E_0^{4,1} & E_0^{5,1} \\
E_0^{-1,0} & E_0^{0,0} & E_0^{1,0} & E_0^{2,0} & E_0^{3,0} & E_0^{4,0} & E_0^{5,0} \\
E_0^{-1,-1} & E_0^{0,-1} & E_0^{1,-1} & E_0^{2,-1} & E_0^{3,-1} & E_0^{4,-1} & E_0^{5,-1}
\end{array}$$

A *spectral sequence* is a sequence of bigraded chain complexes. This is page 1.

$$
\begin{tikzcd}
 & & & & & & & \\
E_1^{-1,2} & E_1^{0,2} \arrow[u] & E_1^{1,2} & E_1^{2,2} & E_1^{3,2} & E_1^{4,2} & E_1^{5,2} & \\
E_1^{-1,1} & E_1^{0,1} \arrow[u, dash] & E_1^{1,1} & E_1^{2,1} & E_1^{3,1} & E_1^{4,1} & E_1^{5,1} & \\
E_1^{-1,0} \arrow[r, dash] & E_1^{0,0} \arrow[u, dash] \arrow[r, dash] & E_1^{1,0} \arrow[r, dash] & E_1^{2,0} \arrow[r, dash] & E_1^{3,0} \arrow[r, dash] & E_1^{4,0} \arrow[r, dash] & E_1^{5,0} \arrow[r] & \\
E_1^{-1,-1} & E_1^{0,-1} \arrow[u, dash] \arrow[d] & E_1^{1,-1} & E_1^{2,-1} & E_1^{3,-1} & E_1^{4,-1} & E_1^{5,-1} & \\
 & & & & & & &
\end{tikzcd}
$$

A *spectral sequence* is a sequence of bigraded chain complexes.
This is page 2.

$$
\begin{tikzcd}
E_2^{-1,2} & E_2^{0,2} \arrow[u] & E_2^{1,2} & E_2^{2,2} & E_2^{3,2} & E_2^{4,2} & E_2^{5,2} \\
E_2^{-1,1} & E_2^{0,1} \arrow[u, dash] & E_2^{1,1} & E_2^{2,1} & E_2^{3,1} & E_2^{4,1} & E_2^{5,1} \\
E_2^{-1,0} \arrow[r, dash] & E_2^{0,0} \arrow[u, dash] \arrow[r, dash] \arrow[d, dash] & E_2^{1,0} \arrow[r, dash] & E_2^{2,0} \arrow[r, dash] & E_2^{3,0} \arrow[r, dash] & E_2^{4,0} \arrow[r, dash] & E_2^{5,0} \arrow[r] & {} \\
E_2^{-1,-1} & E_2^{0,-1} \arrow[d] & E_2^{1,-1} & E_2^{2,-1} & E_2^{3,-1} & E_2^{4,-1} & E_2^{5,-1} \\
& {} & & & & &
\end{tikzcd}
$$

A *spectral sequence* is a sequence of bigraded chain complexes. This is page 3.

$$
\begin{array}{ccccccc}
E_3^{-1,2} & E_3^{0,2} & E_3^{1,2} & E_3^{2,2} & E_3^{3,2} & E_3^{4,2} & E_3^{5,2} \\
E_3^{-1,1} & E_3^{0,1} & E_3^{1,1} & E_3^{2,1} & E_3^{3,1} & E_3^{4,1} & E_3^{5,1} \\
E_3^{-1,0} & E_3^{0,0} & E_3^{1,0} & E_3^{2,0} & E_3^{3,0} & E_3^{4,0} & E_3^{5,0} \\
E_3^{-1,-1} & E_3^{0,-1} & E_3^{1,-1} & E_3^{2,-1} & E_3^{3,-1} & E_3^{4,-1} & E_3^{5,-1}
\end{array}
$$

The *total degree* is the sum of the degrees. Diagonal lines have been drawn to highlight groups of the same total degree.

$$
\begin{array}{ccccccc}
E_0^{0,4} & E_0^{1,4} & E_0^{2,4} & E_0^{3,4} & E_0^{4,4} & E_0^{5,4} & E_0^{6,4} \\
E_0^{0,3} & E_0^{1,3} & E_0^{2,3} & E_0^{3,3} & E_0^{4,3} & E_0^{5,3} & E_0^{6,3} \\
E_0^{0,2} & E_0^{1,2} & E_0^{2,2} & E_0^{3,2} & E_0^{4,2} & E_0^{5,2} & E_0^{6,2} \\
E_0^{0,1} & E_0^{1,1} & E_0^{2,1} & E_0^{3,1} & E_0^{4,1} & E_0^{5,1} & E_0^{6,1} \\
E_0^{0,0} & E_0^{1,0} & E_0^{2,0} & E_0^{3,0} & E_0^{4,0} & E_0^{5,0} & E_0^{6,0}
\end{array}
$$

The *total degree* is the sum of the degrees. Diagonal lines have been drawn to highlight groups of the same total degree.

$$
\begin{array}{ccccccc}
E_1^{0,4} & E_1^{1,4} & E_1^{2,4} & E_1^{3,4} & E_1^{4,4} & E_1^{5,4} & E_1^{6,4} \\
E_1^{0,3} & E_1^{1,3} & E_1^{2,3} & E_1^{3,3} & E_1^{4,3} & E_1^{5,3} & E_1^{6,3} \\
E_1^{0,2} & E_1^{1,2} & E_1^{2,2} & E_1^{3,2} & E_1^{4,2} & E_1^{5,2} & E_1^{6,2} \\
E_1^{0,1} & E_1^{1,1} & E_1^{2,1} & E_1^{3,1} & E_1^{4,1} & E_1^{5,1} & E_1^{6,1} \\
E_1^{0,0} & E_1^{1,0} & E_1^{2,0} & E_1^{3,0} & E_1^{4,0} & E_1^{5,0} & E_1^{6,0}
\end{array}
$$

The *total degree* is the sum of the degrees. Diagonal lines have been drawn to highlight groups of the same total degree.

$$
\begin{array}{ccccccc}
E_2^{0,4} & E_2^{1,4} & E_2^{2,4} & E_2^{3,4} & E_2^{4,4} & E_2^{5,4} & E_2^{6,4} \\
E_2^{0,3} & E_2^{1,3} & E_2^{2,3} & E_2^{3,3} & E_2^{4,3} & E_2^{5,3} & E_2^{6,3} \\
E_2^{0,2} & E_2^{1,2} & E_2^{2,2} & E_2^{3,2} & E_2^{4,2} & E_2^{5,2} & E_2^{6,2} \\
E_2^{0,1} & E_2^{1,1} & E_2^{2,1} & E_2^{3,1} & E_2^{4,1} & E_2^{5,1} & E_2^{6,1} \\
E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & E_2^{3,0} & E_2^{4,0} & E_2^{5,0} & E_2^{6,0}
\end{array}
$$

The *total degree* is the sum of the degrees. Diagonal lines have been drawn to highlight groups of the same total degree.

$$\begin{array}{ccccccc}
E_3^{0,4} & E_3^{1,4} & E_3^{2,4} & E_3^{3,4} & E_3^{4,4} & E_3^{5,4} & E_3^{6,4} \\
E_3^{0,3} & E_3^{1,3} & E_3^{2,3} & E_3^{3,3} & E_3^{4,3} & E_3^{5,3} & E_3^{6,3} \\
E_3^{0,2} & E_3^{1,2} & E_3^{2,2} & E_3^{3,2} & E_3^{4,2} & E_3^{5,2} & E_3^{6,2} \\
E_3^{0,1} & E_3^{1,1} & E_3^{2,1} & E_3^{3,1} & E_3^{4,1} & E_3^{5,1} & E_3^{6,1} \\
E_3^{0,0} & E_3^{1,0} & E_3^{2,0} & E_3^{3,0} & E_3^{4,0} & E_3^{5,0} & E_3^{6,0}
\end{array}$$

- The differential d on $E_r^{*,*}$ has degree $(r, 1-r)$. In other words, we have maps:

$$d|_{E_r^{p,q}} = d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q+1-r}.$$

- The differential d on $E_r^{*,*}$ has degree $(r, 1-r)$. In other words, we have maps:

$$d|_{E_r^{p,q}} = d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q+1-r}.$$

- Note that $d_r^{p,q}$ has total degree 1 since $(p+r)+(q+1-r) = p+q+1$.

- The differential d on $E_r^{*,*}$ has degree $(r, 1-r)$. In other words, we have maps:

$$d|_{E_r^{p,q}} = d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r,q+1-r}.$$

- Note that $d_r^{p,q}$ has total degree 1 since $(p+r)+(q+1-r) = p+q+1$.
- Also $E_{r+1}^{*,*}$ is the homology of the previous page $E_r^{*,*}$. In other words,

$$E_{r+1}^{p,q} = \ker(d_r^{p,q})/\mathrm{im}(d_{r-1}^{p-r,q-1+r}).$$

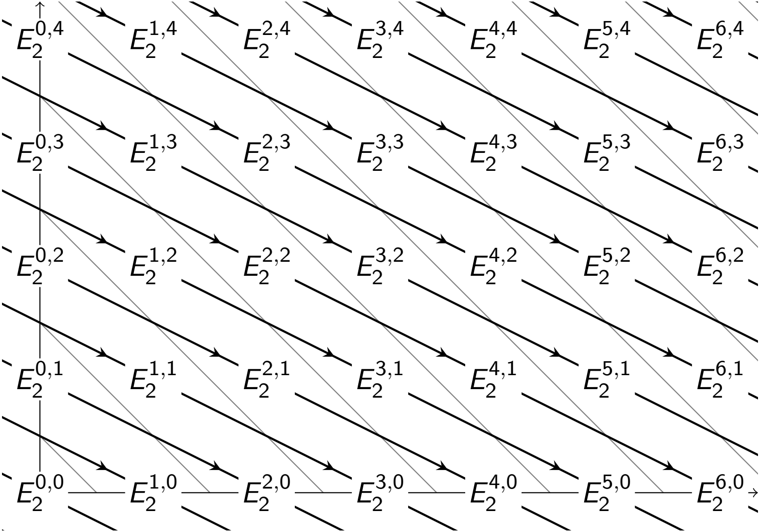
Here the differential has degree $(0, 1-0)$.

$$\begin{tikzcd}
E_0^{0,4} & E_0^{1,4} & E_0^{2,4} & E_0^{3,4} & E_0^{4,4} & E_0^{5,4} & E_0^{6,4} \\
E_0^{0,3} \arrow[u] & E_0^{1,3} \arrow[u] & E_0^{2,3} \arrow[u] & E_0^{3,3} \arrow[u] & E_0^{4,3} \arrow[u] & E_0^{5,3} \arrow[u] & E_0^{6,3} \arrow[u] \\
E_0^{0,2} \arrow[u] & E_0^{1,2} \arrow[u] & E_0^{2,2} \arrow[u] & E_0^{3,2} \arrow[u] & E_0^{4,2} \arrow[u] & E_0^{5,2} \arrow[u] & E_0^{6,2} \arrow[u] \\
E_0^{0,1} \arrow[u] & E_0^{1,1} \arrow[u] & E_0^{2,1} \arrow[u] & E_0^{3,1} \arrow[u] & E_0^{4,1} \arrow[u] & E_0^{5,1} \arrow[u] & E_0^{6,1} \arrow[u] \\
E_0^{0,0} \arrow[u] \arrow[r, dash] & E_0^{1,0} \arrow[u] \arrow[r, dash] & E_0^{2,0} \arrow[u] \arrow[r, dash] & E_0^{3,0} \arrow[u] \arrow[r, dash] & E_0^{4,0} \arrow[u] \arrow[r, dash] & E_0^{5,0} \arrow[u] \arrow[r, dash] & E_0^{6,0} \arrow[u] \arrow[r]
\end{tikzcd}$$

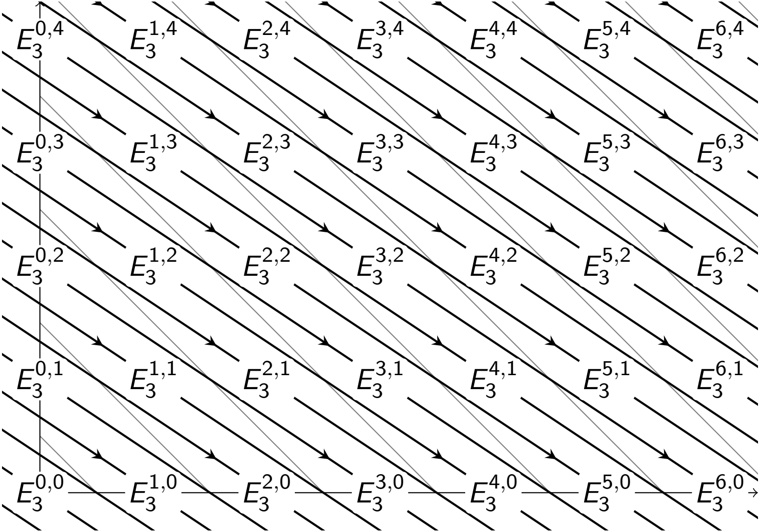
Here the differential has degree $(1, 1-1)$.

$$
\begin{tikzcd}
{} \arrow[r] & E_1^{0,4} \arrow[r] & E_1^{1,4} \arrow[r] & E_1^{2,4} \arrow[r] & E_1^{3,4} \arrow[r] & E_1^{4,4} \arrow[r] & E_1^{5,4} \arrow[r] & E_1^{6,4} \arrow[r] & {} \\
{} \arrow[r] & E_1^{0,3} \arrow[r] & E_1^{1,3} \arrow[r] & E_1^{2,3} \arrow[r] & E_1^{3,3} \arrow[r] & E_1^{4,3} \arrow[r] & E_1^{5,3} \arrow[r] & E_1^{6,3} \arrow[r] & {} \\
{} \arrow[r] & E_1^{0,2} \arrow[r] & E_1^{1,2} \arrow[r] & E_1^{2,2} \arrow[r] & E_1^{3,2} \arrow[r] & E_1^{4,2} \arrow[r] & E_1^{5,2} \arrow[r] & E_1^{6,2} \arrow[r] & {} \\
{} \arrow[r] & E_1^{0,1} \arrow[r] & E_1^{1,1} \arrow[r] & E_1^{2,1} \arrow[r] & E_1^{3,1} \arrow[r] & E_1^{4,1} \arrow[r] & E_1^{5,1} \arrow[r] & E_1^{6,1} \arrow[r] & {} \\
{} \arrow[r] & E_1^{0,0} \arrow[r] & E_1^{1,0} \arrow[r] & E_1^{2,0} \arrow[r] & E_1^{3,0} \arrow[r] & E_1^{4,0} \arrow[r] & E_1^{5,0} \arrow[r] & E_1^{6,0} \arrow[r] & {}
\end{tikzcd}
$$

Here the differential has degree $(2, 1-2)$.



Here the differential has degree $(3, 1-3)$.



What is the E^∞ page?

- It is the set of elements which ‘survive’ forever.

What is the E^∞ page?

- It is the set of elements which 'survive' forever.
- In our case, all the pages $E_r^{p,q}$ for $r > 0$ will be finite dimensional and they decrease in dimension as r increases.

What is the E^∞ page?

- It is the set of elements which ‘survive’ forever.
- In our case, all the pages $E_r^{p,q}$ for $r > 0$ will be finite dimensional and they decrease in dimension as r increases.
- Therefore, for each p, q there is a constant $C^{p,q}$ so that $E_{r+1}^{p,q} = E_r^{p,q}$ for all $r \geq C^{p,q}$. Hence we can define $E_\infty^{p,q}$ to be $E_r^{p,q}$ for $r = C^{p,q}$.

- **Definition:** We say that a spectral sequence $(E_r^{p,q})$ *converges* to a graded group H^* if there is a filtration

$$H^* \supset \cdots F_{-1} \supset F_0 \supset F_1 \supset F_2 \supset \cdots$$

so that

$$E_\infty^{p,q} = F_p \cap H_{p+q} / F_{p+1} \cap H^{p+q}.$$

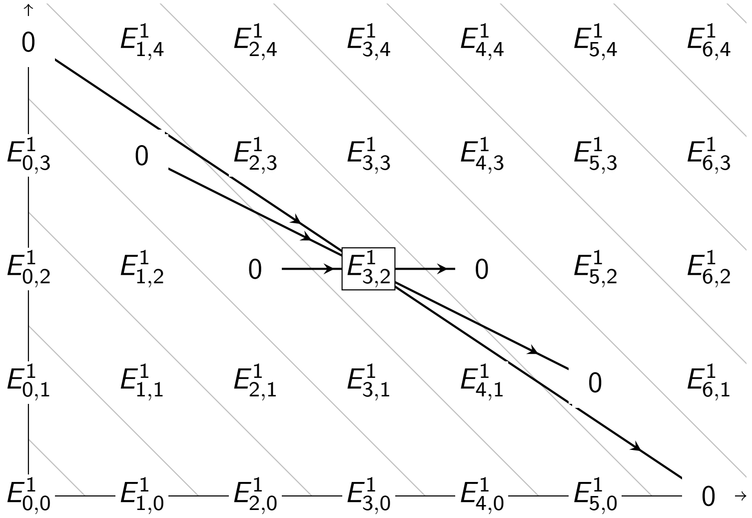
- **Definition:** We say that a spectral sequence $(E_r^{p,q})$ *converges* to a graded group H^* if there is a filtration

$$H^* \supset \cdots F_{-1} \supset F_0 \supset F_1 \supset F_2 \supset \cdots$$

so that

$$E_\infty^{p,q} = F_p \cap H_{p+q}/F_{p+1} \cap H^{p+q}.$$

- In our case the filtration will be nice enough so that if the above spectral sequence converges then $H^n = \oplus_p E_\infty^{p,n-p}$.



Here $H_{3+2} = H_5$ is non-zero.

"... the behavior of this spectral sequence ... is a bit like an Elizabethan drama, full of action, in which the business of each character is to kill at least one other character, so that at the end of the play one has the stage strewn with corpses and only one actor left alive (namely the one who has to speak the last few lines)" - J. F. Adams.

A Spectral Sequence for Symplectic Cohomology.

- We will construct a spectral sequence converging to $SH^*(A)$ (symplectic cohomology of A) where A is a smooth affine variety of dimension n with $c_1(A) = 0$ (there is also a similar spectral sequence when $c_1(A)$ is torsion but we will not focus on that).

A Spectral Sequence for Symplectic Cohomology.

- We will construct a spectral sequence converging to $SH^*(A)$ (symplectic cohomology of A) where A is a smooth affine variety of dimension n with $c_1(A) = 0$ (there is also a similar spectral sequence when $c_1(A)$ is torsion but we will not focus on that).
- Choose a non-zero section κ_A of the *canonical bundle* $K_A \equiv \wedge^n T^* A$ of A.

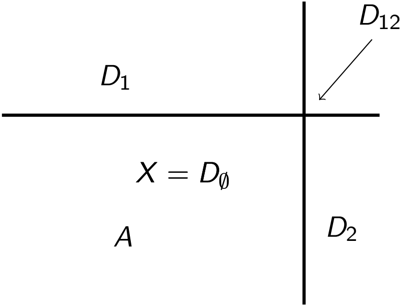
A Spectral Sequence for Symplectic Cohomology.

- We will construct a spectral sequence converging to $SH^*(A)$ (symplectic cohomology of A) where A is a smooth affine variety of dimension n with $c_1(A) = 0$ (there is also a similar spectral sequence when $c_1(A)$ is torsion but we will not focus on that).
- Choose a non-zero section κ_A of the *canonical bundle* $K_A \equiv \wedge^n T^*A$ of A.
- Such a section (up to homotopy) gives $SH^*(A)$ a $\mathbb{Z}$-grading.

- **Definition**: A smooth normal crossing divisor in a smooth projective variety X is a finite union of transversely intersecting smooth complex hypersurfaces $(D_i)_{i \in S}$.

- **Definition**: A smooth normal crossing divisor in a smooth projective variety X is a finite union of transversely intersecting smooth complex hypersurfaces $(D_i)_{i \in S}$.
- **Theorem** (Hironaka) Every smooth affine variety A is isomorphic to $X - \cup_{i \in S} D_i$ for some X, $(D_i)_{i \in S}$ as above. From now on fix this notation.

- **Definition**: A smooth normal crossing divisor in a smooth projective variety X is a finite union of transversely intersecting smooth complex hypersurfaces $(D_i)_{i \in S}$.
- **Theorem** (Hironaka) Every smooth affine variety A is isomorphic to $X - \cup_{i \in S} D_i$ for some X, $(D_i)_{i \in S}$ as above. From now on fix this notation.
- For any $I \subset S$, define $D_I \equiv \cap_{i \in I} D_i$. Here, $D_\emptyset = X$.



$$\begin{gathered} \text{E.g. } A = \mathbb{C}^2 \\ X = \mathbb{CP}^1 \times \mathbb{CP}^1 \\ D_1 = \mathbb{CP}^1 \times \{\infty\} \\ D_2 = \{\infty\} \times \mathbb{CP}^1 \\ D_{12} = \{\infty\} \times \{\infty\} \end{gathered}$$

- We'll assume κ_A is a meromorphic section of the canonical bundle of X which is non-zero along A.

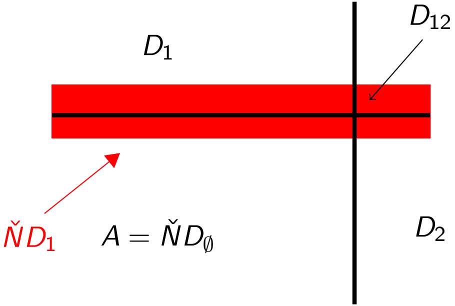
- We'll assume κ_A is a meromorphic section of the canonical bundle of X which is non-zero along A.
- We define the **discrepancy** a_i of D_i to be the order of $\kappa_A^{-1}(0)$ minus the order of $\kappa_A^{-1}(\infty)$ along D_i. I.e. $\kappa_A = z_1^{a_i}$ in some chart $z_1, \cdots, z_n$ satisfying $D_i = \{z_1 = 0\}$.

- We'll assume κ_A is a meromorphic section of the canonical bundle of X which is non-zero along A.
- We define the **discrepancy** a_i of D_i to be the order of $\kappa_A^{-1}(0)$ minus the order of $\kappa_A^{-1}(\infty)$ along D_i. I.e. $\kappa_A = z_1^{a_i}$ in some chart $z_1, \cdots, z_n$ satisfying $D_i = \{z_1 = 0\}$.
- Choose an ample line bundle L on X and a holomorphic section s_A of L so that s_A restricted to A is non-zero and $D = s_A^{-1}(0)$.

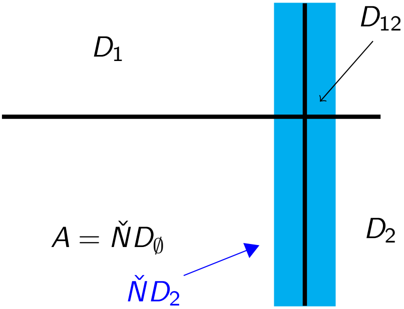
- We'll assume κ_A is a meromorphic section of the canonical bundle of X which is non-zero along A.
- We define the **discrepancy** a_i of D_i to be the order of $\kappa_A^{-1}(0)$ minus the order of $\kappa_A^{-1}(\infty)$ along D_i. I.e. $\kappa_A = z_1^{a_i}$ in some chart $z_1, \cdots, z_n$ satisfying $D_i = \{z_1 = 0\}$.
- Choose an ample line bundle L on X and a holomorphic section s_A of L so that s_A restricted to A is non-zero and $D = s_A^{-1}(0)$.
- We define the **wrapping number** w_i of D_i to be *minus* the order of $s_A^{-1}(0)$ along D_i.

- **Definition:** For each $I \subset S$ let ND_I be a small tubular neighborhood of D_I so that $ND_I \cap D_{I'}$ is a tubular neighborhood of $D_{I \cup I'}$ for all $I' \subset S$. Also ∂ND_I should intersect $D_{I'}$ transversally for all $I' \subset S$.

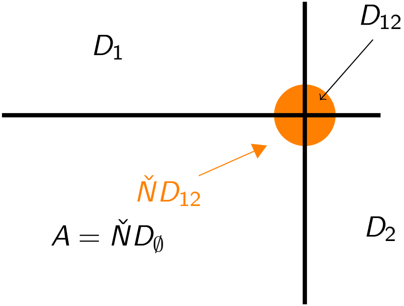
- **Definition:** For each $I \subset S$ let ND_I be a small tubular neighborhood of D_I so that $ND_I \cap D_{I'}$ is a tubular neighborhood of $D_{I \cup I'}$ for all $I' \subset S$. Also ∂ND_I should intersect $D_{I'}$ transversally for all $I' \subset S$.
- Define $\check{N}D_I \equiv ND_I - \cup_{i \in S} D_i$. This as a bundle over $\check{V}_I \equiv D_I - \cup_{i \in S-I} D_i$ with fiber a product of punctured disks.



- **Definition:** For each $I \subset S$ let ND_I be a small tubular neighborhood of D_I so that $ND_I \cap D_{I'}$ is a tubular neighborhood of $D_{I \cup I'}$ for all $I' \subset S$. Also ∂ND_I should intersect $D_{I'}$ transversally for all $I' \subset S$.
- Define $\check{N}D_I \equiv ND_I - \cup_{i \in S} D_i$. This as a bundle over $\check{V}_I \equiv D_I - \cup_{i \in S-I} D_i$ with fiber a product of punctured disks.



- **Definition:** For each $I \subset S$ let ND_I be a small tubular neighborhood of D_I so that $ND_I \cap D_{I'}$ is a tubular neighborhood of $D_{I \cup I'}$ for all $I' \subset S$. Also ∂ND_I should intersect $D_{I'}$ transversally for all $I' \subset S$.
- Define $\check{N}D_I \equiv ND_I - \cup_{i \in S} D_i$. This as a bundle over $\check{V}_I \equiv D_I - \cup_{i \in S-I} D_i$ with fiber a product of punctured disks.



Theorem (M - in progress):
There is a spectral sequence converging to $SH^*(A)$ with E_1 page

$$E_1^{p,q} = \bigoplus_{\{(k_i)\in\mathbb{N}^S \ : \ \sum_i k_i w_i = p\}} H^{p+q-2(\sum_i k_i(a_i+1))}(\check{N}D_{I_{(k_i)}})$$

where $\mathbb{N}^S$ is the set of tuples of non-negative integers indexed by S and $I_{(k_i)} = \{i \in S \ : \ k_i \neq 0\}$.

- There is a similar spectral sequence for $SH^*_{>0}(A)$ where we sum over everything except the term corresponding to $(0) \in \mathbb{N}^S$.

- There is a similar spectral sequence for $SH^*_{>0}(A)$ where we sum over everything except the term corresponding to $(0) \in \mathbb{N}^S$.
- If c_1 is torsion then κ_A is a section of $K_A^{\otimes r}$ and the discrepancies a_i are now defined to be the order of $\kappa_A^{-1}(0)$ minus the order of $\kappa_A^{-1}(\infty)$ along D_i divided by r. The associated spectral sequence is identical but the pages could potentially have entries with non-integer p, q since a_i may not be an integer. The differentials have the same gradings.

- There is a similar spectral sequence for $SH^*_{>0}(A)$ where we sum over everything except the term corresponding to $(0) \in \mathbb{N}^S$.
- If c_1 is torsion then κ_A is a section of $K_A^{\otimes r}$ and the discrepancies a_i are now defined to be the order of $\kappa_A^{-1}(0)$ minus the order of $\kappa_A^{-1}(\infty)$ along D_i divided by r. The associated spectral sequence is identical but the pages could potentially have entries with non-integer p, q since a_i may not be an integer. The differentials have the same gradings.
- The future work of Diogo-Lisi and Ganatra-Pomerleano hopefully should give better descriptions of the differentials in some cases.

Other Grading Conventions

- There are other grading conventions.
- You might need to replace (p, q) with $(-p, n-q)$ for symplectic *homology* and your spectral sequence differentials will go in the other direction.

Sanity Check

- $X = \mathbb{CP}^n$, $D_1 = \mathbb{CP}^{n-1}$ and $A = \mathbb{C}^n$.

Sanity Check

- $X = \mathbb{CP}^n$, $D_1 = \mathbb{CP}^{n-1}$ and $A = \mathbb{C}^n$.
- $w_1 = -1$ and $a_1 = -n - 1$.

Sanity Check

- $X = \mathbb{CP}^n$, $D_1 = \mathbb{CP}^{n-1}$ and $A = \mathbb{C}^n$.
- $w_1 = -1$ and $a_1 = -n - 1$.
- $H^*(\check{N} D_1) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \text{ or } 2n-1 \\ 0 & \text{otherwise.} \end{cases}$.

Sanity Check

- $X = \mathbb{CP}^n$, $D_1 = \mathbb{CP}^{n-1}$ and $A = \mathbb{C}^n$.
- $w_1 = -1$ and $a_1 = -n-1$.
- $H^*(\check{N}D_1) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \text{ or } 2n-1 \\ 0 & \text{otherwise.} \end{cases}$.
- $H^*(\check{N}D_\emptyset) = H^*(A) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ 0 & \text{otherwise} \end{cases}$.

0	0	0	0	0	0	0
0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}$
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}$	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0
0	0	0	0	0	0	0

Case: $n = 3$

$E_2^{p,q} = 0$ for all p, q

$SH^*(A) = 0$

Example

- Let X be a smooth degree 5 hypersurface in $\mathbb{CP}^3$ and let D_1 be the intersection of X with a generic degree 1 hypersurface and $A = X - D_1$.

Example

- Let X be a smooth degree 5 hypersurface in $\mathbb{CP}^3$ and let D_1 be the intersection of X with a generic degree 1 hypersurface and $A = X - D_1$.
- $w_1 = -1$ and $a_1 = 1$.

Example

- Let X be a smooth degree 5 hypersurface in $\mathbb{CP}^3$ and let D_1 be the intersection of X with a generic degree 1 hypersurface and $A = X - D_1$.
- $w_1 = -1$ and $a_1 = 1$.
- $H^*(\check{N}D_1) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \text{ or } 3 \\ \mathbb{Z}^{12} & \text{if } * = 1 \text{ or } 2 \\ 0 & \text{otherwise.} \end{cases}$.

Example

- Let X be a smooth degree 5 hypersurface in $\mathbb{CP}^3$ and let D_1 be the intersection of X with a generic degree 1 hypersurface and $A = X - D_1$.
- $w_1 = -1$ and $a_1 = 1$.
- $H^*(\check{N}D_1) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \text{ or } 3 \\ \mathbb{Z}^{12} & \text{if } * = 1 \text{ or } 2 \\ 0 & \text{otherwise.} \end{cases}$.
- $H^*(\check{N}D_\emptyset) = H^*(A) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z}^{64} & \text{if } * = 2 \\ 0 & \text{otherwise} \end{cases}$.

Example

- Let X be a smooth degree 5 hypersurface in $\mathbb{CP}^3$ and let D_1 be the intersection of X with a generic degree 1 hypersurface and $A = X - D_1$.
- $w_1 = -1$ and $a_1 = 1$.
- $H^*(\check{N}D_1) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \text{ or } 3 \\ \mathbb{Z}^{12} & \text{if } * = 1 \text{ or } 2 \\ 0 & \text{otherwise.} \end{cases}$.
- $H^*(\check{N}D_\emptyset) = H^*(A) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z}^{64} & \text{if } * = 2 \\ 0 & \text{otherwise} \end{cases}$.
- Computations using ideas from Milnor's paper "On simply connected 4-manifolds". See also https://amathew.wordpress.com/2012/03/05/the-cohomology-of-projective-hypersurfaces/

$$
\begin{array}{ccccccc}
 & & \mathbb{Z}^{12} & & & & \uparrow \\
0 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbb{Z}^{12} & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbb{Z}^{12} & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbb{Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbb{Z}^{12} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbb{Z}^{12} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbb{Z} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbb{Z}^{12} & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbb{Z}^{12} & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbb{Z} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z}^{64} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 - & 0 - & 0 - & 0 - & 0 - & 0 - & \mathbb{Z} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Therefore

$$
SH^*(A) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z}^{64} & \text{if } * = 2 \\ \mathbb{Z} & \text{if } * > 3 \text{ and } * = 0 \text{ or } 3 \mod 4 \\ \mathbb{Z}^{12} & \text{if } * > 3 \text{ and } * = 1 \text{ or } 2 \mod 4 \\ 0 & \text{otherwise.} \end{cases}
$$

Example with two divisors

- X be a smooth degree 6 hypersurface in $\mathbb{CP}^3$, D_1, D_2 are generic degree 1 hypersurfaces and $A = X - D_1 - D_2$.

Example with two divisors

- X be a smooth degree 6 hypersurface in $\mathbb{CP}^3$, D_1, D_2 are generic degree 1 hypersurfaces and $A = X - D_1 - D_2$.
- $w_1 = w_2 = -1$ and $a_1 = a_2 = 1$.

Example with two divisors

- X be a smooth degree 6 hypersurface in $\mathbb{CP}^3$, D_1, D_2 are generic degree 1 hypersurfaces and $A = X - D_1 - D_2$.
- $w_1 = w_2 = -1$ and $a_1 = a_2 = 1$.
- $H^*(\check{N}D_{12}) = \begin{cases} \mathbb{Z}^6 & \text{if } * = 0 \text{ or } 2 \\ \mathbb{Z}^{12} & \text{if } * = 1 \\ 0 & \text{otherwise.} \end{cases}$.

Example with two divisors

- X be a smooth degree 6 hypersurface in $\mathbb{CP}^3$, D_1, D_2 are generic degree 1 hypersurfaces and $A = X - D_1 - D_2$.
- $w_1 = w_2 = -1$ and $a_1 = a_2 = 1$.
- $H^*(\check{N}D_{12}) = \begin{cases} \mathbb{Z}^6 & \text{if } * = 0 \text{ or } 2 \\ \mathbb{Z}^{12} & \text{if } * = 1 \\ 0 & \text{otherwise.} \end{cases}$.
- $H^*(\check{N}D_1) = H^*(\check{N}D_2) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z}^{26} & \text{if } * = 1 \\ \mathbb{Z}^{25} & \text{if } * = 2 \\ 0 & \text{otherwise.} \end{cases}$.

Example with two divisors

- X be a smooth degree 6 hypersurface in $\mathbb{CP}^3$, D_1, D_2 are generic degree 1 hypersurfaces and $A = X - D_1 - D_2$.
- $w_1 = w_2 = -1$ and $a_1 = a_2 = 1$.
- $H^*(\check{N}D_{12}) = \begin{cases} \mathbb{Z}^6 & \text{if } * = 0 \text{ or } 2 \\ \mathbb{Z}^{12} & \text{if } * = 1 \\ 0 & \text{otherwise.} \end{cases}$.
- $H^*(\check{N}D_1) = H^*(\check{N}D_2) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z}^{26} & \text{if } * = 1 \\ \mathbb{Z}^{25} & \text{if } * = 2 \\ 0 & \text{otherwise.} \end{cases}$.
- $H^*(\check{N}D_\emptyset) = H^*(A) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \text{ or } 1 \\ \mathbb{Z}^{150} & \text{if } * = 2 \\ 0 & \text{otherwise} \quad . \end{cases}$.

$$
\begin{array}{ccccccc}
 & & \mathbb{Z}^{[illegible]} & & & & \uparrow \\
0 & 0 & \mathbb{Z}^{20} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbb{Z}^{62} & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbb{Z}^{76} & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbb{Z}^{14} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbb{Z}^{56} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbb{Z}^{64} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbb{Z}^{8} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbb{Z}^{50} & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbb{Z}^{52} & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbb{Z}^{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z}^{150} \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbb{Z} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
$$

Therefore

$$
SH^*(A) = \begin{cases}
\mathbb{Z} & \text{if } * = 0 \text{ or } 1 \\
\mathbb{Z}^{150} & \text{if } * = 2 \\
\mathbb{Z}^{2+3(*-4)/2} & \text{if } * > 3 \text{ and } * = 0 \mod 4 \\
\mathbb{Z}^{52+3(*-5)} & \text{if } * > 3 \text{ and } * = 1 \mod 4 \\
\mathbb{Z}^{50+3(*-6)/2} & \text{if } * > 3 \text{ and } * = 2 \mod 4 \\
0 & \text{otherwise.}
\end{cases}
$$

Weinstein Conjecture

- **Weinstein conjecture**: Every contact form has a Reeb orbit.

Weinstein Conjecture

- **Weinstein conjecture**: Every contact form has a Reeb orbit.
- **Definition**: A cooriented contact manifold (C, ξ) satisfies the Weinstein conjecture if every contact form α compatible with ξ has a Reeb orbit.

Weinstein Conjecture

- **Weinstein conjecture**: Every contact form has a Reeb orbit.
- **Definition**: A cooriented contact manifold (C, ξ) satisfies the Weinstein conjecture if every contact form α compatible with ξ has a Reeb orbit.
- Which contact manifolds satisfy the Weinstein conjecture?

- Recall that positive symplectic cohomology $SH^*_{>0}(M)$ of a Liouville domain M has a chain complex freely generated by two copies of each Reeb orbit on ∂M. In other words, we do not consider critical points of the Hamiltonian in the interior.

- Recall that positive symplectic cohomology $SH^*_{>0}(M)$ of a Liouville domain M has a chain complex freely generated by two copies of each Reeb orbit on ∂M. In other words, we do not consider critical points of the Hamiltonian in the interior.
- **Definition**: M satisfies the **algebraic Weinstein conjecture** if $SH^*_{>0}(M) \neq 0$.

- Recall that positive symplectic cohomology $SH^*_{>0}(M)$ of a Liouville domain M has a chain complex freely generated by two copies of each Reeb orbit on ∂M. In other words, we do not consider critical points of the Hamiltonian in the interior.
- **Definition**: M satisfies the **algebraic Weinstein conjecture** if $SH^*_{>0}(M) \neq 0$.
- **Lemma:** If M satisfies the algebraic Weinstein conjecture then ∂M satisfies the Weinstein conjecture.

- Recall that positive symplectic cohomology $SH^*_{>0}(M)$ of a Liouville domain M has a chain complex freely generated by two copies of each Reeb orbit on ∂M. In other words, we do not consider critical points of the Hamiltonian in the interior.
- **Definition**: M satisfies the **algebraic Weinstein conjecture** if $SH^*_{>0}(M) \neq 0$.
- **Lemma:** If M satisfies the algebraic Weinstein conjecture then ∂M satisfies the Weinstein conjecture.
- **Question:** Which smooth affine varieties satisfy the algebraic Weinstein conjecture?

- $X =$ smooth projective variety and $A = X - \cup_i D_i$ where $(D_i)_{i \in S}$ is a smooth normal crossing divisor.

- X = smooth projective variety and $A = X - \cup_i D_i$ where $(D_i)_{i \in S}$ is a smooth normal crossing divisor.
- **Theorem:** Suppose that the discrepancy a_i of D_i is ≤ -1 for all $i \in S$. Then A satisfies the algebraic Weinstein conjecture.

- X = smooth projective variety and $A = X - \cup_i D_i$ where $(D_i)_{i \in S}$ is a smooth normal crossing divisor.
- **Theorem:** Suppose that the discrepancy a_i of D_i is ≤ -1 for all $i \in S$. Then A satisfies the algebraic Weinstein conjecture.
- **Proof of the main Theorem**:

$$\begin{array}{c|cccc}
0 & E_1^{*,*} & E_1^{*,*} & E_1^{*,*} & E_1^{*,*} \\
0 & E_1^{*,*} & E_1^{*,*} & E_1^{*,*} & E_1^{*,*} \\
0 & 0 & E_1^{*,*} & 0 & E_1^{*,*} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}$$

This is the lowest non-zero E_1 term on the lowest diagonal.

This term exists since $a_i \leq -1, \ \forall i$ and it survives to the E_∞ page since all differentials connecting this term have source or target 0. $\square$

Additional Structure

- For many important varieties (e.g log Calabi-Yau varieties), the spectral sequence does not help us compute $SH^*(A)$ as the differentials may not be 0. Also we wish to compute $SH^*(A)$ as an algebra with the pair of pants product.

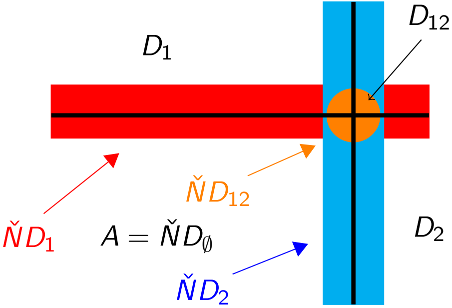
Additional Structure

- For many important varieties (e.g log Calabi-Yau varieties), the spectral sequence does not help us compute $SH^*(A)$ as the differentials may not be 0. Also we wish to compute $SH^*(A)$ as an algebra with the pair of pants product.
- A spectral sequence $E_*^{*,*}$ is a **spectral sequence of algebras** if each page $E_r^{*,*}$ is a differential bigraded algebra so that the product structure on $E_{r+1}^{*,*}$ is induced by the product structure on $E_r^{*,*}$ for each r.

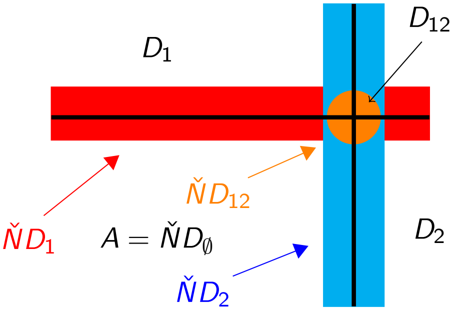
Additional Structure

- For many important varieties (e.g log Calabi-Yau varieties), the spectral sequence does not help us compute $SH^*(A)$ as the differentials may not be 0. Also we wish to compute $SH^*(A)$ as an algebra with the pair of pants product.
- A spectral sequence $E_*^{*,*}$ is a **spectral sequence of algebras** if each page $E_r^{*,*}$ is a differential bigraded algebra so that the product structure on $E_{r+1}^{*,*}$ is induced by the product structure on $E_r^{*,*}$ for each r.
- Convergence is defined in the same way, except that the filtration has to respect the product structure on the algebra H_*.

- Assume: $\check{N}D_{I'} \subset \check{N}D_I$ for all $I \subset I'$.

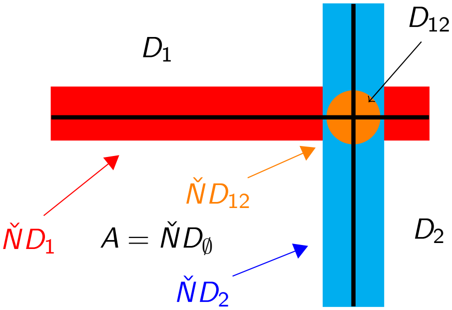


- Assume: $\check{N}D_{I'} \subset \check{N}D_I$ for all $I \subset I'$.
- Let $\iota_{I',I} : \check{N}D_{I'} \to \check{N}D_I$ be the natural inclusion map.



- Assume: $\check{N}D_{I'} \subset \check{N}D_I$ for all $I \subset I'$.
- Let $\iota_{I',I} : \check{N}D_{I'} \to \check{N}D_I$ be the natural inclusion map.
- For all $I, J \subset S$, define:

$$\begin{aligned} P_{IJ} : H^*(\check{N}D_I) \otimes H^*(\check{N}D_J) &\longrightarrow H^*(\check{N}D_{I\cup J}) \\ a \otimes b &\longrightarrow \iota^*_{I\cup J,I}a \ \cup \ \iota^*_{I\cup J,J}b. \end{aligned}$$



Conjecture/Theorem? (Ganatra-Pomerleano, work in progress) The spectral sequence above is in fact a spectral sequence of algebras converging to $SH^*(A)$ with the pair of pants product. The product structure

$$E_1^{p,q} \otimes E_1^{p',q'} \longrightarrow E_1^{p+p',q+q'}$$

on the E_1 page

$$E_1^{p,q} = \bigoplus_{\left\{(k_i)\in\mathbb{N}^S \,:\, \sum_i k_i w_i = p\right\}} H^{p+q-2(\sum_i k_i(a_i+1))}(\check{N}D_{I_{(k_i)}}).$$

is induced by the maps P_{IJ} above.

- **Definition**. An **isolated family of Reeb orbits** B **of** (C, α) **of length** l is a subset $B \subset C$ consisting of Reeb orbits of length l so that there is a neighborhood $\mathcal{N}$ of B so that there are no Reeb orbits in $\mathcal{N}$ of length in $[l - \epsilon, l + \epsilon]$ for some small $\epsilon > 0$.

- **Definition**. An **isolated family of Reeb orbits** B **of** (C, α) **of length** l is a subset $B \subset C$ consisting of Reeb orbits of length l so that there is a neighborhood $\mathcal{N}$ of B so that there are no Reeb orbits in $\mathcal{N}$ of length in $[l - \epsilon, l + \epsilon]$ for some small $\epsilon > 0$.
- **Definition** (c.f. Kirwan 1985): A **minimally degenerate subset of length** l is a closed subset $B \subset C$ so that there is a function $f : C \longrightarrow (0, \infty)$ and a submanifold $N \subset C$ (possibly with boundary) satisfying

- **Definition**. An **isolated family of Reeb orbits** B **of** (C, α) **of length** l is a subset $B \subset C$ consisting of Reeb orbits of length l so that there is a neighborhood $\mathcal{N}$ of B so that there are no Reeb orbits in $\mathcal{N}$ of length in $[l - \epsilon, l + \epsilon]$ for some small $\epsilon > 0$.
- **Definition** (c.f. Kirwan 1985): A **minimally degenerate subset of length** l is a closed subset $B \subset C$ so that there is a function $f : C \longrightarrow (0, \infty)$ and a submanifold $N \subset C$ (possibly with boundary) satisfying
  1. $B \subset N$,

- **Definition**. An **isolated family of Reeb orbits** B **of** (C, α) **of length** l is a subset $B \subset C$ consisting of Reeb orbits of length l so that there is a neighborhood $\mathcal{N}$ of B so that there are no Reeb orbits in $\mathcal{N}$ of length in $[l - \epsilon, l + \epsilon]$ for some small $\epsilon > 0$.
- **Definition** (c.f. Kirwan 1985): A **minimally degenerate subset of length** l is a closed subset $B \subset C$ so that there is a function $f : C \longrightarrow (0, \infty)$ and a submanifold $N \subset C$ (possibly with boundary) satisfying
  1. $B \subset N$,
  2. B is an isolated family of Reeb orbits of length l,

- **Definition**. An **isolated family of Reeb orbits** B **of** (C, α) **of length** l is a subset $B \subset C$ consisting of Reeb orbits of length l so that there is a neighborhood $\mathcal{N}$ of B so that there are no Reeb orbits in $\mathcal{N}$ of length in $[l - \epsilon, l + \epsilon]$ for some small $\epsilon > 0$.
- **Definition** (c.f. Kirwan 1985): A **minimally degenerate subset of length** l is a closed subset $B \subset C$ so that there is a function $f : C \longrightarrow (0, \infty)$ and a submanifold $N \subset C$ (possibly with boundary) satisfying
  1. $B \subset N$,
  2. B is an isolated family of Reeb orbits of length l,
  3. N is a Morse-Bott submanifold of $(C, f\alpha)$,

- **Definition**. An **isolated family of Reeb orbits** B **of** (C, α) **of length** l is a subset $B \subset C$ consisting of Reeb orbits of length l so that there is a neighborhood $\mathcal{N}$ of B so that there are no Reeb orbits in $\mathcal{N}$ of length in $[l - \epsilon, l + \epsilon]$ for some small $\epsilon > 0$.
- **Definition** (c.f. Kirwan 1985): A **minimally degenerate subset of length** l is a closed subset $B \subset C$ so that there is a function $f : C \longrightarrow (0, \infty)$ and a submanifold $N \subset C$ (possibly with boundary) satisfying
  1. $B \subset N$,
  2. B is an isolated family of Reeb orbits of length l,
  3. N is a Morse-Bott submanifold of $(C, f\alpha)$,
  4. $f^{-1}(1) = B$ and 1 is the maximum of f.

- **Definition**. An **isolated family of Reeb orbits** B **of** (C, α) **of length** l is a subset $B \subset C$ consisting of Reeb orbits of length l so that there is a neighborhood $\mathcal{N}$ of B so that there are no Reeb orbits in $\mathcal{N}$ of length in $[l - \epsilon, l + \epsilon]$ for some small $\epsilon > 0$.
- **Definition** (c.f. Kirwan 1985): A **minimally degenerate subset of length** l is a closed subset $B \subset C$ so that there is a function $f : C \longrightarrow (0, \infty)$ and a submanifold $N \subset C$ (possibly with boundary) satisfying
  1. $B \subset N$,
  2. B is an isolated family of Reeb orbits of length l,
  3. N is a Morse-Bott submanifold of $(C, f\alpha)$,
  4. $f^{-1}(1) = B$ and 1 is the maximum of f.
- **Definition** : A **minimally degenerate contact pair** is a contact pair (C, α) so that every periodic orbit is contained inside a minimally degenerate subset.

- **Proposition (not proven yet):** Suppose that the boundary of (M, θ) is a minimally degenerate contact pair and let $(B_k^j)_{k \in \mathbb{N}, j \in I_k}$ be the set of all of the minimally degenerate subsets so that
  1. they are connected,
  2. I_k is a finite set for all $k \in \mathbb{N}$,
  3. the length of B_k^j is the length of $B_k^{j'}$ for all $j, j' \in I_k$ and these lengths tend to infinity as $k \to \infty$ and
  4. the length of B_k^j is less than the length of $B_{k+1}^{j'}$ for all $k \in \mathbb{N}$, $j \in I_k$ and $j' \in I_{k+1}$.

  Then there is spectral sequence converging to $SH^{p+q}(M)$ with E^1 page

  $$E_1^{p,q} = \bigoplus_{j \in I_p} \check{H}^{n-p-q+CZ(B_p)}(B_p^j, \mathcal{L}_{B_p^j})$$

  where $\mathcal{L}_{B_p^j}$ is a certain local coefficient system.

- Our spectral sequence will be obtained by constructing an appropriate minimally degenerate contact form on the boundary of our Liouville domain.

- Our spectral sequence will be obtained by constructing an appropriate minimally degenerate contact form on the boundary of our Liouville domain.
- In order to find a such a boundary, we need to construct a symplectically nice neighborhood of the divisor in question (resolution divisor or compactifying divisor).

- Our spectral sequence will be obtained by constructing an appropriate minimally degenerate contact form on the boundary of our Liouville domain.
- In order to find a such a boundary, we need to construct a symplectically nice neighborhood of the divisor in question (resolution divisor or compactifying divisor).
- This requires a purely symplectic notion of divisor. See 1011.2542 and work of M-Tehrani-Zinger.

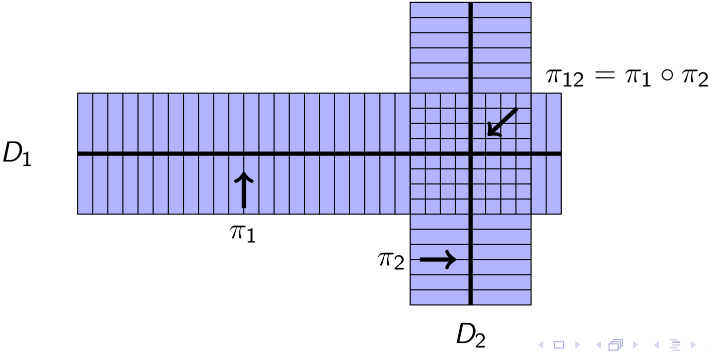
- Our spectral sequence will be obtained by constructing an appropriate minimally degenerate contact form on the boundary of our Liouville domain.
- In order to find a such a boundary, we need to construct a symplectically nice neighborhood of the divisor in question (resolution divisor or compactifying divisor).
- This requires a purely symplectic notion of divisor. See 1011.2542 and work of M-Tehrani-Zinger.
- The corresponding minimally degenerate subsets are submanifolds with boundary and corners associated to the strata of the compactification divisor.

1. Let (M, ω) be a symplectic $2n$-manifold.
2. **Definition**: A **symplectic SNC divisor** is a finite collection $(D_i)_{i \in S}$ of transversally intersecting submanifolds so that

$$D_I := \cap_{i \in I} D_i$$

is symplectic for all $i \in I$ and so that the symplectic orientation $\omega^{2n-2|I|}|_{D_I}$ of D_I coincides with its associated intersection orientation coming from $\cap_{i \in I} D_i$.

- A **regularization** of a symplectic SNC divisor $(D_i)_{i \in S}$ inside (X, ω) consists of tubular fibrations $(\pi_I)_{I \subset S}$ of $(D_I)_{I \subset S}$ with symplectic fibers so that
  1. $\pi_{I_1 \cup I_2} = \pi_{I_1} \circ \pi_{I_2}$ on their common domain of definition for all $I_1, I_2 \subset S$ and
  2. the fibers of π_I are symplectomorphic to a product $\prod_{i \in I} \mathbb{D}(\epsilon)$ of ϵ disks and the associated symplectic connection has parallel transport maps rotating these disks giving us a $U(1)^{|I|}$ structure group.
  3. There should also be a particular almost complex structure but we won't need this.



- **Theorem** M (2011), M-Tehrani-Zinger (2014): Every symplectic SNC divisor is isotopic through symplectic SNC divisors to one which admits a regularization.

- **Theorem** M (2011), M-Tehrani-Zinger (2014): Every symplectic SNC divisor is isotopic through symplectic SNC divisors to one which admits a regularization.
- The proof first involves proving the Theorem in the linear case first and then using a Moser argument to extend this linear argument to the general non-linear case.

Construction of Spectral Sequence (extremely vague sketch)

- Let X be our projective variety compactifying our affine variety A by a smooth normal crossing divisor D.
- Deform D so that it admits a regularization as above.
- Construct a natural Liouville vector field W compatible with this regularization.
- If $(r_i)_{i \in S}$ are the natural radial coordinates then our contact hypersurface transverse to W is a regular level set of $\sum_{i \in S} f(r_i)$.