Floer Cohomology and Birational Geometry.

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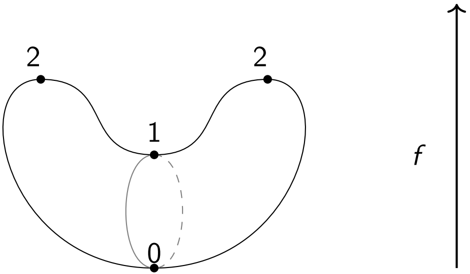
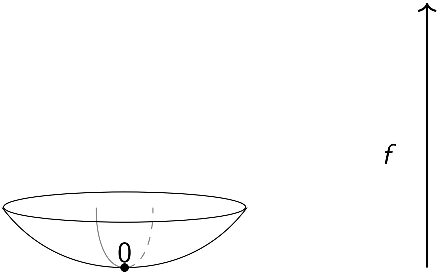
What is Floer Cohomology?

- *Floer (co)homology* is an infinite dimensional version of Morse (co)homology.
- So, what is Morse homology?
- A *Morse function* is a smooth function $f : M \longrightarrow \mathbb{R}$ with the property that there is an atlas of charts where f looks like a linear function or a non-degenerate quadratic function plus a constant in each such chart.
- A Morse function then gives a cellular decomposition of M.

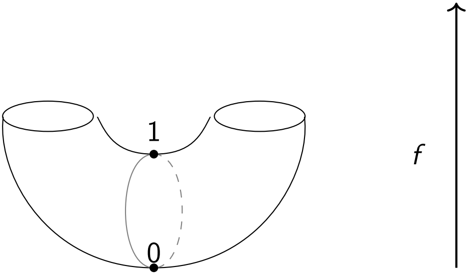
Example:



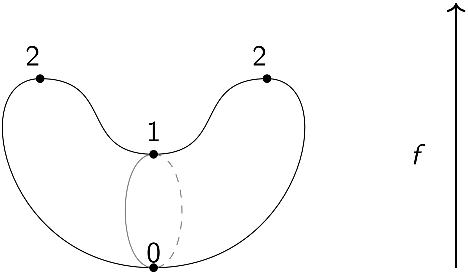
0-cell $\simeq$



0-cell $\cup$ 1-cell $\simeq$



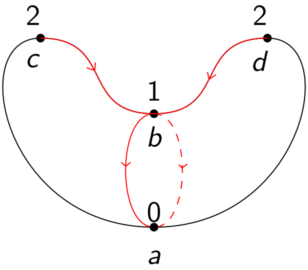
0-cell $\cup$ 1-cell $\simeq$
$\cup$ 2-cell $\cup$ 2-cell



- Hence we can calculate the homology of M using f as follows:
- The chain complex is the free $\mathbb{Z}$-module

$$\mathbb{Z}\langle p_1, \cdots, p_k \rangle$$

  generated by critical points of f and graded by Morse index.
- Now we choose a generic Riemannian metric g on M.
- The differential is a $k \times k$ matrix whose coefficient corresponding to (p_i, p_j) is the number of gradient flowlines of f connecting p_i and p_j (counted with sign).



Here

$$C_0 = \mathbb{Z}\langle a\rangle,\ \ C_1 = \mathbb{Z}\langle b\rangle,\ \ C_2 = \mathbb{Z}\langle c, d\rangle$$

We have

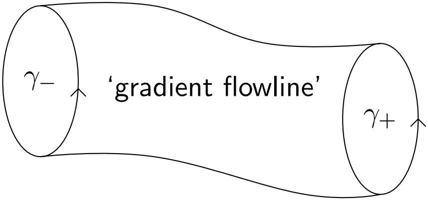
$$\partial a = 0, \qquad \partial b = a - a = 0, \qquad \partial c = b, \qquad \partial d = b$$

Hence $H_*(S^2) = \begin{cases} \mathbb{Z} & \text{if } * = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$

Floer Cohomology

- *Floer Cohomology* is an infinite dimensional version of Morse homology.
- I will describe an example, *Hamiltonian Floer Cohomology* $HF^*(H)$ of a Hamiltonian H on a symplectic manifold M.
- Our infinite dimensional 'manifold' is the *free loopspace* of a symplectic manifold M. I.e. the space of maps from S^1 to M.
- The 'Morse function' in this case is a function called the *action functional*, whose critical points correspond to periodic solutions of Hamilton's equations.
- A 'gradient flowline' connecting such critical points is a map from a cylinder to M whose ends limit to these periodic solutions and which satisfies a Cauchy-Riemann like PDE.

The Floer differential counts such cylinders connecting periodic solutions γ_- and γ_+ of Hamilton's equations.



Cohomological McKay Correspondence.

- Let $G \subset SU(n)$ be a finite subgroup and consider the quotient $\mathbb{C}^n/G$.
- A *resolution* of $\mathbb{C}^n/G$ is a proper birational morphism

$$\pi : Y \longrightarrow \mathbb{C}^n/G$$

  from a smooth variety Y.
- It is *crepant* if $c_1(Y) = 0$.

Cohomological McKay Correspondence.

- Let $\mathbb{K}$ be a field.
- **Conjecture** (Reid).
  Cohomological McKay Correspondence over $\mathbb{K}$.
  For any crepant resolution Y as above, there is a basis of $H^*(Y; \mathbb{K})$ consisting of conjugacy classes of G.
- **Theorem** (Batyrev 1998): This is true when $\mathbb{K} = \mathbb{Q}$.
- One way of proving this theorem is via an Euler characteristic style argument called *motivic integration* developed by Kontsevich.

Floer Theoretic Proof

- Suppose the resolution Y is Kähler (and hence a symplectic manifold).
- Consider the Hamiltonian $H = \pi^* r^4$ on our crepant resolution Y where r is the radial coordinate on $\mathbb{C}^n/G$.
- There is a natural map $C^*(Y) \longrightarrow CF^*(H)$ between the chain complexes for $H^*(Y;\mathbb{K})$ and $HF^*(H)$ respectively and we define $HF^*_+(H)$ to be the cone of this chain map.

Floer Theoretic Proof

- **Theorem** (M., Ritter). Suppose $\mathbb{K}$ is a field of characteristic $> |G|$ and G acts freely away from 0. Then
  1. $HF^*_+(H)$ is isomorphic to $H^*(Y;\mathbb{K})$ and
  2. $HF^*_+(H)$ is generated by conjugacy classes of G.
- **Corollary**. The cohomological McKay correspondence holds over $\mathbb{K}$.

Brief Idea of Proof

- To show that $HF^*_+(H) = H^*(Y; \mathbb{K})$, one shows $HF^*(H) = 0$ and we do this by deforming H inside a compact subset making all the 1-periodic orbits have arbitrarily large index.
- To show $HF^*_+(H)$ is generated by conjugacy classes of G, one exploits the fact that the corresponding Floer cochain complex is filtered by action (our 'Morse function').
- Such a filtration gives a spectral sequence which can be used to compute $HF^*_+(H)$ in terms of conjugacy classes of G.

Birational Calabi-Yau Manifolds

- By a *Calabi-Yau manifold* I will mean a smooth projective variety with trivial first Chern class.
- Two Calabi-Yau manifolds are *birational* if they are isomorphic outside a codimension 1-subvariety of each.
- **Question:** What properties do such Calabi-Yau manifolds have in common?

- **Lemma**. If X and $\widehat{X}$ are birational Calabi-Yau manifolds then there are codimension 2 subvarieties $V \subset X$ and $\widehat{V} \subset \widehat{X}$ so that $X - V \cong \widehat{X} - \widehat{V}$.
- **Corollary**. We have a canonical isomorphism $H_2(X) \cong H_2(\widehat{X})$.

- **Theorem** (Batyrev). Birational Calabi-Yau manifolds have the same Betti numbers.
- Again, there is a proof using Motivic integration.
- However there are examples of birational Calabi-Yau manifolds whose cohomology *rings* are different.

Quantum Cup Product

- Define the *Novikov Ring* $\Lambda_{\mathbb{K}}^{\omega} = \left\{ \sum_{i\in\mathbb{N}} a_i t^{\beta_i} \;\middle|\; a_i \in \mathbb{K},\ \beta_i \in H_2(X;\mathbb{Z}),\ \omega(\beta_i) \to \infty \right\}$ where ω is the symplectic form on our Calabi-Yau manifold X.
- Choose a homogenous basis $A_1, \cdots, A_k \in H^*(X;\mathbb{K})$ together with the dual basis $\widehat{A}_1, \cdots, \widehat{A}_k \in H^*(X;\mathbb{K})$ with respect to the cup product pairing $(\eta, \nu) \to \int_X \eta \cup \nu$.

Quantum Cup Product

- We define *small quantum cohomology* $QH^*(X;\Lambda_{\mathbb{K}}^{\omega})$ to be the unique $\Lambda_{\mathbb{K}}^{\omega}$-algebra structure on $H^*(X;\Lambda_{\mathbb{K}}^{\omega})$ whose product $\star_X$ satisfies:

$$A_i \star_X A_j = \sum_{\beta \in H_2(X;\mathbb{Z})} \sum_{m=1}^{k} GW_{0,3}^{X,\beta}(A_i, A_j, A_m)\widehat{A}_m t^{\beta}$$

  where $GW_{0,3}^{X,\beta}(A_i, A_j, A_m)$ is the number of holomorphic maps $\mathbb{P}^1 \longrightarrow X$ representing β passing through cycles Poincaré dual to A_i, A_j and A_k.
- If we only considered the constant terms (i.e. coefficients of t^0), then this would be the usual cup product.
- There is also *big quantum cohomology* which involves counts of curves passing through arbitrary numbers of cycles.

- **Conjecture** (Morrison, Ruan). Any two birational Calabi-Yau manifolds have the same (small or big) quantum cohomology groups up to analytic continuation.
- **Theorem** (Li-Ruan). This is true in dimension 3.
- **Theorem** (Iwao, Lee, Lin, Qu, Wang). This is true if the birational transform is a composition of ordinary flops.

- Let X and $\widehat{X}$ be birational Calabi-Yau manifolds.
- Let ω and $\widehat{\omega}$ be Kähler forms on X and $\widehat{X}$ and $\Lambda_{\mathbb{K}}^{\omega}$, $\Lambda_{\mathbb{K}}^{\widehat{\omega}}$ the corresponding Novikov rings.
- Using the identification $H_2(X) = H_2(\widehat{X})$, we can take the intersection $\Lambda_{\mathbb{K}}^{\omega,\widehat{\omega}} = \Lambda_{\mathbb{K}}^{\omega} \cap \Lambda_{\mathbb{K}}^{\widehat{\omega}}$.
- **Theorem** (M.) There exists a graded $\Lambda_{\mathbb{K}}^{\omega,\widehat{\omega}}$-algebra Z together with algebra isomorphisms:

$$Z \otimes_{\Lambda_{\mathbb{K}}^{\omega,\widehat{\omega}}} \Lambda_{\mathbb{K}}^{\omega} \cong QH^*(X;\Lambda_{\mathbb{K}}^{\omega}), \quad Z \otimes_{\Lambda_{\mathbb{K}}^{\omega,\widehat{\omega}}} \Lambda_{\mathbb{K}}^{\widehat{\omega}} \cong QH^*(\widehat{X};\Lambda_{\mathbb{K}}^{\widehat{\omega}}).$$

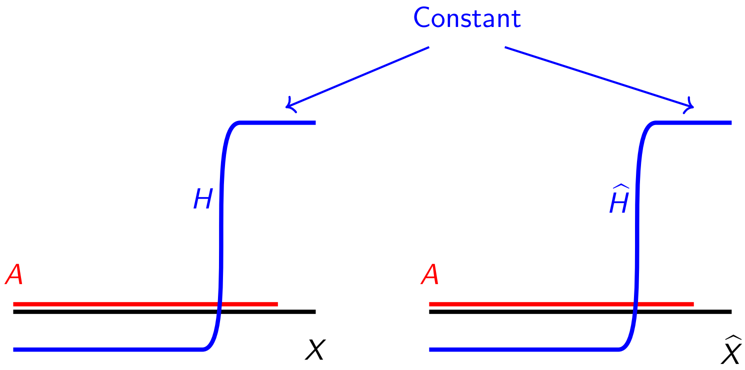
- The key idea here is to use Hamiltonian Floer cohomology
- This has a natural product called the *pair of pants product*
- For any Hamiltonian H on X, we have that $HF^*(H) \cong QH^*(X; \mathbb{K})$ and similarly for $\widehat{X}$.
- Therefore it is sufficient to find appropriate Hamiltonians H and $\widehat{H}$ on X and $\widehat{H}$ together with an isomorphism $HF^*(H) \cong HF^*(\widehat{H})$.

Basic Idea of the Proof.

- Recall we have two birational CY manifolds X and $\widehat{X}$ so that $X - V \cong \widehat{X} - \widehat{V}$ for some codimension ≥ 2 subvarieties V and $\widehat{V}$.
- Choose a common affine subvariety $A \subset X - V$, $A \subset \widehat{X} - \widehat{V}$.
- Next we use the fact that Hamiltonian Floer cohomology is isomorphic as an algebra to small quantum cohomology.

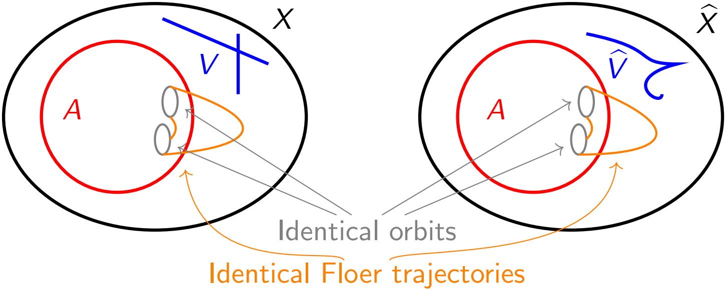
Basic Idea of the Proof

- We now put two Hamiltonians H and $\widehat{H}$ on X and $\widehat{X}$ respectively with graphs as follows:



Basic Idea of the Proof

- Now suppose that we can ignore the constant orbits near $X - A$ and $\widehat{X} - \widehat{A}$ respectively.
- Then generators of our Floer complexes are identical.
- Since V and $\widehat{V}$ are of codimension 2, the Floer trajectories joining these generators avoid V and $\widehat{V}$ and hence or identical. Hence our Hamiltonian Floer algebras are identical.



Basic Idea of the Proof

- Now one issue is that you cannot just ignore a family of constant orbits.
- One must take a direct limit of these Hamiltonians H and $\widehat{H}$ that get bigger and bigger and restrict to certain subcomplexes.
- With this setup, you can ignore the constant orbits near $X - A$ and $\widehat{X} - A$ respectively. Here, we also need to use the Calabi-Yau condition.