# Floer Cohomology and Arc Spaces

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- **Observation** (Seidel 2001): Floer Cohomology of an isolated hypersurface singularity is similar to a certain space of arcs.
- Very roughly, the **arc space** of a singularity is the space of holomorphic maps from the unit disk passing through that singularity. Actually this is called the **short arc space**.
- We will be only interested in jets of such maps.
- However, my personal opinion is that the entire arc space is important if one wishes to study other more complicated Floer groups.

- **General goal:** To understand this relationship between Floer theory and arc spaces better.
- Another hope is that it makes it easier to compute Floer groups.
- This talk will be about some work in progress.
- Advertising spiel for algebraic geometers: You too can prove results in symplectic geometry, without doing much symplectic geometry!

# Milnor Monodromy Map

- Let $f : \mathbb{C}^n \longrightarrow \mathbb{C}$ be a polynomial with an isolated singularity at 0 satisfying $f(0) = 0$.
- Let $S_\epsilon \subset \mathbb{C}^n$ be the sphere of small radius $\epsilon > 0$.
- The **Milnor map** is defined to be the symplectic fibration

$$\frac{f}{|f|} : S_\epsilon - f^{-1}(0) \longrightarrow S^1.$$

# Milnor Monodromy Map

- The **Milnor fiber** $M_f$ **of** $f$ is a fiber of this map.
- The **Milnor monodromy map**

$$\phi_f : M_f \longrightarrow M_f$$

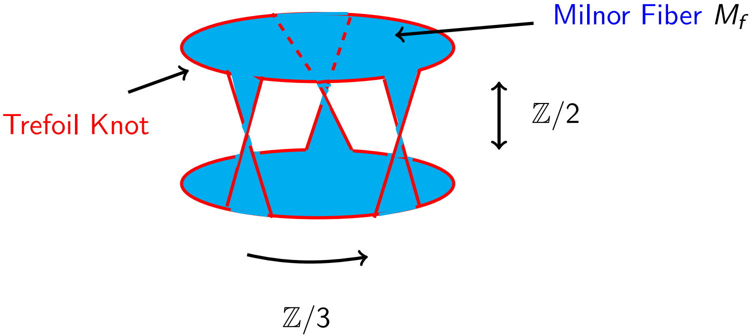
is the monodromy of the Milnor fibration around $S^1$.

- **Fact:** The closure of $M_f$ in $S_\epsilon$ is a Liouville domain and $\phi_f$ extends to an exact symplectomorphism of this Liouville domain (after modifying the fibration slightly).
- Recall a **Liouville domain** is a pair $(M, \theta)$ where
  - $M$ is a compact manifold with boundary.
  - $d\theta$ is a symplectic form.
  - The unique vector field $X_\theta$ satisfying $i_{X_\theta} d\theta = \theta$ points outwards along $\partial M$.

- $S_\epsilon$ has a natural contact structure $\xi_f := TS_\epsilon \cap J_0 TS_\epsilon$ where $J_0$ is the standard complex structure on $\mathbb{C}^n$.
- **Fact:** $f^{-1}(0)$ is a contact submanifold of $S_\epsilon$. This is called the **link** of $f$.

# Example

- Consider $f(x, y) = x^2 + y^3$.
- The **link** $S_\epsilon \cap f^{-1}(0)$ is the trefoil knot.
- The Milnor fiber is a torus with one boundary component.




$\phi_f$ generates $\mathbb{Z}/6 = \mathbb{Z}/2 \times \mathbb{Z}/3$ action
(modulo some boundary rotation factor)

# Floer Cohomology

- Now let $\phi : M \longrightarrow M$ be an exact symplectomorphism of a Liouville domain $M$ with Liouville form $\theta$.
- **Floer Cohomology** $HF^*(\phi)$ is defined as follows:
- The generators are fixed points of $\phi$, or equivalently, constant sections of the mapping torus

$$T_\phi = [0,1] \times M / \sim, \quad ((1,x) \sim (0, \phi_f(x))).$$

- The differential counts certain holomorphic sections of $\mathbb{R} \times T_\phi$.
- *Technical remark:* Grading is given by minus Conley-Zehnder index, with trivialization induced by $\mathbb{C}^n$.

- **Theorem:** Suppose $\check{f} : \mathbb{C}^n \longrightarrow \mathbb{C}$ is another polynomial with isolated singularity at 0. Let $\phi_{\check{f}}$ be the Milnor monodromy map of $\check{f}$. Suppose that there is a contactomorphism of $S_\epsilon$ sending the link of $f$ to the link of $\check{f}$. Then

$$HF^*(\phi_f^d) \cong HF^*(\phi_{\check{f}}^d)$$

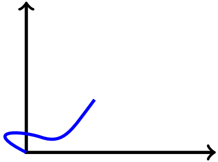
  for each $d \in \mathbb{N}$.
- In other words, Floer cohomology is an invariant of the link as a contact submanifold.

# Contact Loci

- Let $\mathbb{D} \subset \mathbb{C}$ be the unit disk.
- The ***d*th jet space** $\mathrm{Jet}^d(\mathbb{C}^n)|_0$ is the variety of $d$-jets of holomorphic maps $v : \mathbb{D} \to \mathbb{C}$ satisfying $v(0) = 0$.
- More concretely it is the affine space of $n$-tuples of degree $d$ polynomials whose constant coefficient vanishes:

$$\left\{ u(t) := a_d t^d + a_{d-1} t^{d-1} + \cdots + a_1 t \ : \ a_1, \cdots, a_d \in \mathbb{C}^n \right\}$$

$$\cong (t\mathbb{C}[t]/(t^{d+1}))^n \cong \mathbb{C}^{dn}.$$



# Contact Loci

- The $d$**th contact locus** $\chi_d(f)$ of our polynomial $f$ is the subspace:

$$\{u(t) \in \mathrm{Jet}^d(\mathbb{C}^n)|_0 \ : \ f(u(t)) = t^d \in \mathrm{Jet}^d(\mathbb{C})|_0\}.$$

In other words, the subspace of $d$-jets that map via $f$ to the $d$-jet of the map $z \to z^d$.

- More concretely: it is the space:

$$\chi_d(f) := \left\{ u(t) = \sum_{i=1}^{d} a_i t^i \in \mathbb{C}^n[t] \ : \ f(u(t)) = t^d \bmod t^{d+1} \right\}.$$

- Morally, it is the space of $d$-jets of holomorphic maps $\mathbb{D} \longrightarrow \mathbb{C}$ whose boundary 'wraps' around the singularity $d$ times.

# Example

**Example:** Suppose $f(x,y) = x^2 + y^3 \in \mathbb{C}[x,y]$.

Then $\chi_2(f) = \left\{ u(t) = \mathrm{Jet}^2(\mathbb{C}^2)|_0 \; : \; f(u(t)) = t^2 \mod t^3 \right\} =$

$$\left\{ \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} t^2 + \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} t : (a_{21}t^2 + a_{11}t)^2 + (a_{22}t^2 + a_{12}t)^3 = t^2 \mod t^3 \right\}$$

$$= \left\{ (a_{11}, a_{12}, a_{21}, a_{22}) \in \mathbb{C}^4 \; : \; a_{11}^2 = 1 \right\}$$

$$\cong \mathbb{C}^3 \sqcup \mathbb{C}^3.$$

# A Conjecture

**Conjecture:** (Nero Budur, Javier Fernández de Bobadilla, Quy Thuong Lê, Hong Duc Nguyen):

$$HF^*(\phi_f^d) \cong H_c^{*+2nd+n-1}(\chi_d(f)).$$

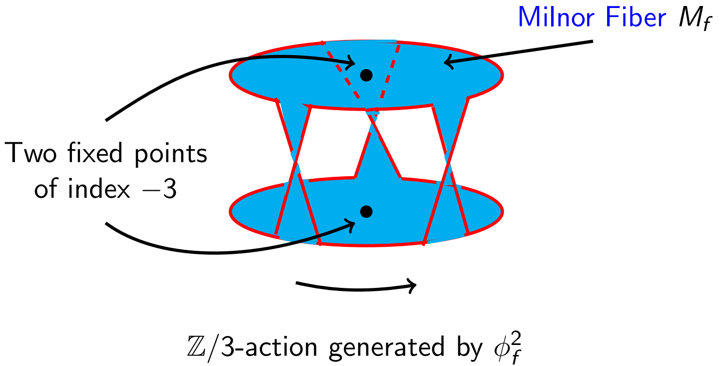
**Theorem:** (BdBLN). Suppose $f = f_m + f_{m+1} + \cdots$ where $f_i$ is a homogenous polynomial of degree $i$ for each $i$. Then the groups above with $d = m$ are isomorphic to $H_c^*(F)$, where $F$ is the Milnor fiber of $f_m$. Hence for $d = m$ the conjecture is true.

**Corollary**: $H_c^*(F)$ is a contact invariant of the link (viewed as a contact submanifold of $S_\epsilon$).

**Theorem** (M, In progress). The conjecture above is true in general.

# Our Example.

- Let us consider this conjecture for $f(x, y) = x^2 + y^3$ where $d = 2$.
- We computed $\chi_2(f) = \mathbb{C}^3 \sqcup \mathbb{C}^3$ and hence $H_c^*(\chi_2(f))$ is equal to $\mathbb{Z} \oplus \mathbb{Z}$ in degree 6 and 0 otherwise.




- $HF^*(\phi_f^2) = \mathbb{Z} \oplus \mathbb{Z}$ if $* = 2$ and 0 otherwise. Hence the Conjecture is true in this example.

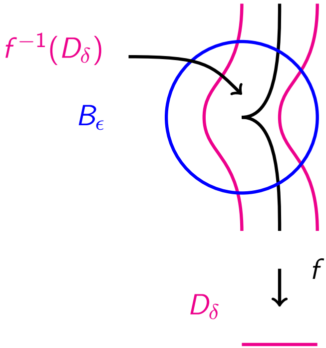
# Proof Idea

- We will first construct a natural morphism (called a PSS map):
$$\text{ev} : HF^*(\phi_f^d) \longrightarrow H_c^{*+2nd+n-1}(\chi_d(f)).$$
- On the chain level, this will be a count of holomorphic disks whose boundary limits to an orbit of the mapping torus and so that the $d$-jet of this disk sweeps out a cycle in a thickening of $\chi_d(f)$ inside $\text{Jet}^d(f)|_0$.
- This map is similar in spirit to the log PSS map defined by Ganatra and Pomerleano.

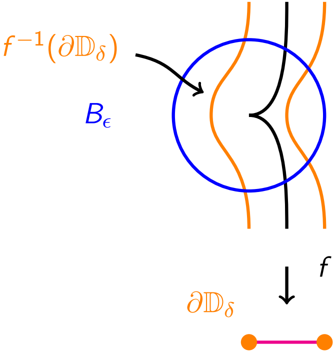
We start with the singular fibration:

$$f : f^{-1}(\mathbb{D}_\delta) \cap B_\epsilon \longrightarrow \mathbb{D}_\epsilon$$

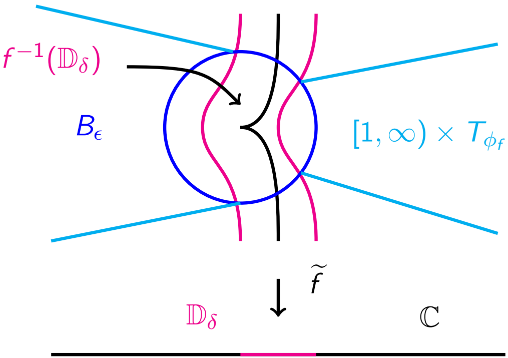
where $0 < \delta \ll \epsilon \ll 1$ and where $\mathbb{D}_\delta \subset \mathbb{C}$ is the disk of radius $\delta$ and $B_\epsilon \subset \mathbb{C}^n$ is the closed ball of radius $\epsilon$.



Now $f^{-1}(\partial\mathbb{D}_\delta) \cap B_\epsilon$ (up to deformation) is the mapping torus $T_{\phi_f}$ of $\phi_f$.



We now glue a cylindrical end $[1, \infty) \times T_{\phi_f}$ to this fibration giving us a large space $E_{\widetilde{f}}$. We also extend the map $f$ to $\widetilde{f} : E_{\widetilde{f}} \longrightarrow \mathbb{C}$ in a natural way.



At the chain level our PSS map:

$$\mathrm{ev} : HF^*(\phi_f^d) \longrightarrow H_c^{*+2nd+n-1}(\chi_d(f)).$$

is defined as follows: We need to compute $\mathrm{ev}(p)$ for a fixed point $p$ of $\phi_f^d$. Choose the unique constant $d$-fold multi section of the mapping torus $T_{\phi_f} = [0,1] \times M_f / \sim$ passing through $p$. This is a map:

$$\gamma : S^1 \longrightarrow T_{\phi_f}$$

with image $[0,1] \times \{p\}/\sim$ so that its composition with the natural map:

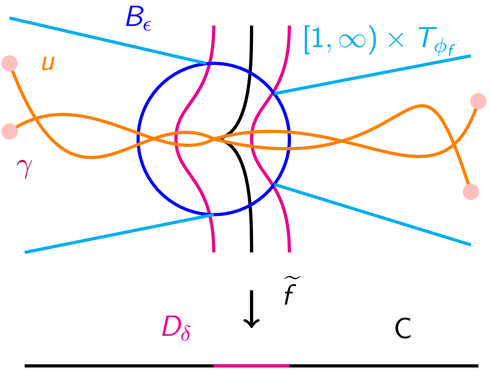
$$T_{\phi_f} \longrightarrow S^1$$

is the $d$-fold covering map:

$$S^1 \longrightarrow S^1, \; z \longrightarrow z^d.$$

Let $\mathcal{M}(p)$ be the space of $J$-holomorphic maps $u : \mathbb{C} \to E_{\widetilde{f}}$ so that

1. $\lim_{r\to\infty} u(re^{it})$ is $\gamma(e^{it})$.
2. The $d$-jet of $\widetilde{f} \circ u : \mathbb{C} \longrightarrow \mathbb{C}$ is the $d$-jet of $z \longrightarrow z^d$.
3. $u(0) = 0$.



We now have a natural evaluation map:

$$\mathrm{ev} : \mathcal{M}(p) \longrightarrow \chi_d(f)$$

sending $u$ to its $d$-jet at 0. Hence ev represents a locally finite homology cycle. Hence it represents an element of $C_c^*(\chi_d(f))$ (modulo some details).

One issue is that $\chi_d(f)$ is singular, however it is a subvariety of $\mathrm{Jet}^d(\mathbb{C}^n)|_0$ and so you can consider some larger cycle in $\mathrm{Jet}^d(\mathbb{C}^n)|_0$ and restrict to $\chi_d(f)$.

A standard PSS style gluing and compactness argument ensures that ev is a chain map.

To show that ev is an isomorphism, we actually need to work in a small neighborhood of the preimage of $\chi_d(f)$ inside $\mathrm{Jet}^l(\mathbb{C}^n)|_0$ for some large $l \geq d$.

However in this talk we will suppress this detail.

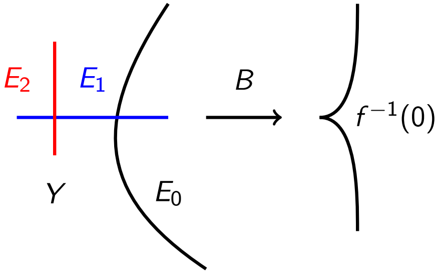
**Key idea of the proof:** Construct filtrations $F$ and $F'$ on the chain complexes computing $H_c^{*+2nd+n-1}(\chi_d(f))$ and $HF^*(\phi_f^d)$ and show that our PSS map ev

1. respects these filtrations and
2. induces an isomorphism on the associated graded filtration.

These filtrations come from spectral sequences. One by Budur, de Bobadilla, Lê, Nguyen (1911.08213) and one by by myself (1608.07541).

Both filtrations are constructed using a *resolution* of $f$. By Hiranoka resolution of singularities, we can find a holomorphic map $B : Y \longrightarrow \mathbb{C}^n$ so that

1. $B$ is an isomorphism onto $\mathbb{C}^n - 0$.
2. $B^{-1}(f^{-1}(0))$ is a union of transversally intersecting complex hypersurfaces $E_0, \cdots, E_l$.
3. There is an integral Kähler form $\omega_Y$ on $Y$ so that $\omega_Y = d\theta_Y$ away from $B^{-1}(0)$.



# Construction of $F$

Our filtration on $H_c^{*+2nd+n-1}(\chi_d(f))$ will be induced by a function

$$F : \chi_d(f) \longrightarrow \mathbb{R}.$$

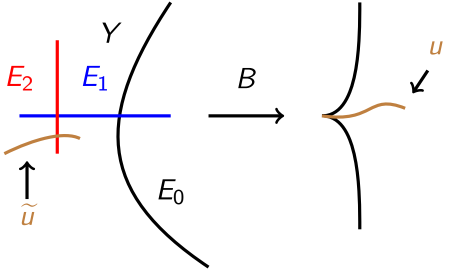
For simplicity, we will assume $d$ is large.

Let $u : \mathbb{D} \longrightarrow \mathbb{C}^n$ be a holomorphic map representing an element $\eta$ of $\chi_d(f)$. We define

$$F(\eta) := \lim_{r \to 0} \int_0^{2\pi} \frac{d}{dt} u(re^{it})(\theta_Y) dt.$$

This induces the filtration $F$ on the chain complex computing $H_c^{*+2nd+n-1}(\chi_d(f))$.

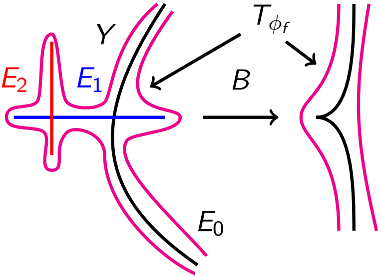
The key point here is that, by the removable singularity theorem, $u$ admits a lift $\widetilde{u} : \mathbb{D} \longrightarrow Y$. Such an arc intersects a stratum of $\cup_i E_i$. The value of $F$ is determined by this stratum.




Also if I have a family of arcs $u_t$, $t \in [0,1]$, with corresponding lifts $(\widetilde{u}_t)_{t\in[0,1]}$, then $\widetilde{u}_t$ Gromov converges to $\widetilde{u}_0$ possibly with a bubble tree attached. The positive energy of this bubble tree ensures that $F$ gives us a filtration on the chain complex.

# The filtration on $CF^*(\phi_f^d)$

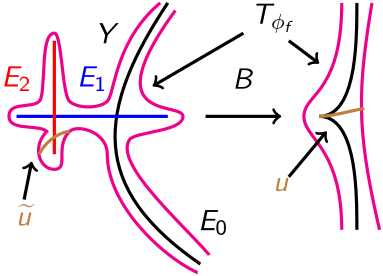
We now will describe the filtration $F'$ on the chain complex $CF^*(\phi_f^d)$. First of all, we deform the mapping torus $T_{\phi_f}$ so that it is 'close' to $\cup_i E_i$.




The filtration $F'$ is now given by integrating $\theta_Y$ over each orbit $\gamma : S^1 \longrightarrow T_{\phi_f}$. These orbits come in families corresponding to each stratum of $\cup_i E_i$.

By Stokes' theorem, we see that the PSS map ev respects the filtrations $F$ and $F'$.

Also since $T_{\phi_f}$ is 'close' to $\cup_i E_i$, the holomorphic multisections defining the PSS map ev have small energy. Hence $F'(\gamma)$ and $F(\mathrm{ev}(\gamma))$ are very close. Therefore to show that ev induces a quasi-isomorphism on the associated graded complexes, one only needs to consider low energy holomorphic disks. Such low energy disks have very small diameter in the resolution $Y$.




In the diagram above, you can see the family of disks $\widetilde{u}$ intersecting $E_2 - E_1$ transversally at 0. The compactly supported cohomology of such a family of disks is part of the $E^1$ page of the spectral sequence associated to $F$. There is a family of orbits going around $E_2$ near $E_2 - E_1$ whose local Floer cohomology is the same group. This local Floer group gives the corresponding part of the $E_1$ page for the $F'$ filtration.

# Further Directions

- There is a pants product (and also many other Floer theoretic operations) on $\oplus_{d \in \mathbb{N}} HF^*(\phi_f^d)$. What do they correspond to on the arc space?
- Ganatra and Pomerleano constructed a spectral sequence computing symplectic cohomology of an affine variety from a smooth normal crossing compactification. What happens if the compactification is no longer smooth normal crossing? The hope is that one can build the $E^1$ page from various spaces of (low energy) arcs.

- Suppose I have an isolated singularity (which is not necessarily a hypersurface singularity). Can I compute (full) contact homology using the arc space? This is a difficult question since such groups can be infinite dimensional in every degree (both positive and negative). However, can one build some kind of spectrum from the (short) arc space using these groups. Note that one has to consider short arcs by work of Kollár (I.e. one has to go beyond jets).