Homework 1 (MAT 562)

(1) Let (V, Ω) be a symplectic vector space and let $W \subset V$ be a linear symplectic subspace. Show that $W \oplus W^{\perp,\Omega} = V$.

Definition: A linear subspace W of a symplectic vector space (V, Ω) is called *isotropic* if $\Omega|_W = 0$. A subspace W is called *coisotropic* if $\Omega|_{W^{\perp,\Omega}} = 0$.

(2) Suppose that W is isotropic or coisotropic. Show that there is a symplectic basis $P_1, \cdots, P_n, Q_1, \cdots, P_n$ of V so that W is the span the first k vectors of this basis where $k = \dim(W)$.

(3) Let (V, Ω) be a symplectic vector space of dimension 4. Construct two 2-dimensional linear symplectic subspaces $W_1, W_2 \subset V$ so that $W_1 \oplus W_2 = V$ and so that the natural orientation on V induced by $\Omega \wedge \Omega$ is *minus* the orientation on $W_1 \oplus W_2$ induced by $\pi_1^*(\Omega|_{W_1}) \wedge \pi_2^*(\Omega|_{W_2})$ where $\pi_1 : W_1 \oplus W_2 \longrightarrow W_1$ and $\pi_2 : W_1 \oplus W_2 \longrightarrow W_2$ are the natural projection maps.

(4) Let $S^{2n+1} \subset \mathbb{C}^{n+1}$ be the unit sphere and let the group $U(1)$ of units in $\mathbb{C}$ act diagonally on $\mathbb{C}^{n+1}$ by multiplication.
   (a) Show that $\omega_{\text{std}}|_{S^{2n+1}}$ is $U(1)$ invariant and that the orbits of this group action are one dimensional submanifolds of S^{2n+1}.
   (b) Show that each vector v tangent to one of these orbits satisfies $\omega_{\text{std}}(v, w) = 0$ for each vector $w \in TS^{2n+1}$ at the same point.
   (c) Hence show there is a unique 2-form ω_{FS} (called the *Fubini-Study form*) on $\mathbb{C}P^n = S^{2n+1}/U(1)$ whose pullback is ω_{std}.

(5) Let (M_1, ω_1), (M_2, ω_2) be symplectic manifolds and let $\Phi : M_1 \longrightarrow M_2$ be a diffeomorphism. Show that Φ is a symplectomorphism if and only if its graph if $M_1 \times M_2$ is Lagrangian with respect to the symplectic form $-\pi_1^*\omega_1 + \pi_2^*\omega_2$ where $\pi_i : M_1 \times M_2 \longrightarrow M_i$, $i = 1, 2$ is the natural projection map.