Homework 3

(1) Show that a 2-form ω on a smooth manifold is non-degenerate if and only if ω^m is a volume form for some $m \in \mathbb{N}$.

(2) Show that a submanifold of a symplectic manifold is Lagrangian if and only if it is both isotropic and coisotropic.

(3) Show that a submanifold V is symplectic if and only if $TV \oplus TV^\perp = TM|_V$.

(4) Construct two 2-dimensional linear symplectic subspaces W_1, W_2 of $\mathbb{R}^4$ so that
   (a) $W_1 \oplus W_2 = \mathbb{R}^4$,
   (b) W_1 and W_2 are symplectic submanifolds of $(\mathbb{R}^4, \omega_{\text{std}})$
   (c) and the orientation on $W_1 \oplus W_2$ induced by $\pi_1^*(\omega_{\text{std}}|_{W_1}) \wedge \pi_2^*(\omega_{\text{std}}|_{W_2})$ is minus the natural orientation induced by $\omega_{\text{std}} \wedge \omega_{\text{std}}$ on $\mathbb{R}^4$ where $\pi_1 : W_1 \oplus W_2 \longrightarrow W_1$ and $\pi_2 : W_1 \oplus W_2 \longrightarrow W_2$ are the natural projection maps.

(5) Let V be a closed submanifold of a smooth closed manifold M and let ω, ω' two symplectic forms on M agreeing along $TM|_V$. Show that there is a diffeomorphism $\phi : M \to M$ fixing V and a neighborhood U of V so that $\phi^*(\omega|_U) = \omega'|_{\phi^{-1}(U)}$.

(6) Let $S^{2n+1} \subset \mathbb{C}^{n+1}$ be the unit sphere and let the group $U(1)$ of units in $\mathbb{C}$ act diagonally on $\mathbb{C}^{n+1}$ by multiplication.
   (a) Show that $\omega_{\text{std}}|_{S^{2n+1}}$ is $U(1)$ invariant and that the orbits of this smooth group action are one dimensional submanifolds of S^{2n+1}.
   (b) Show that each vector $v \in T_p S^{2n+1}$, $p \in S^{2n+1}$, tangent to one of these orbits satisfies $\omega_{\text{std}}(v, w) = 0$ for each vector $w \in T_p S^{2n+1}$.
   (c) Hence show there is a unique 2-form ω_{FS} (called the *Fubini-Study form*) on $\mathbb{C}P^n = S^{2n+1}/U(1)$ whose pullback is ω_{std}.

(7) Show that the Hamiltonian symplectomorphism group of a symplectic manifold M acts transitively on M.