Homework 2

(1) Find the critical points of the following functional:

$$\mathcal{P} \to \mathbb{R},\ f \to \int_0^1 xf'(x) - (f'(x))^2 dx.$$

where $\mathcal{P}$ is the space of smooth functions $f : [0,1] \to \mathbb{R}$ satisfying $f(0) = 0$ and $f(1) = 1$.

(2) (Arnold page 58). Give examples functionals

$$\Phi(\gamma) = \int_{t_0}^{t_1} L(\mathbf{x}, \dot{\mathbf{x}}, t)dt$$

that have many critical points and others that have none.

(3) (Arnold page 59). Let x_0, x_1 be points in $\mathbb{R}^n$ and let Ω_{x_0,x_1} be the space of smooth paths $\gamma : [0,1] \longrightarrow \mathbb{R}^n$ joining x_0 and x_1. Consider the functional

$$\Phi : \Omega_{x_0,x_1} \longrightarrow \mathbb{R}, \quad \Phi(\gamma) := \int_0^1 |\dot{\gamma}(t)|^2 dt.$$

Show that it has exactly one extremal value and that this is the unique minimum of Φ.

(4) (Arnold, page 65). Let

$$f : \mathbb{R}^n \to \mathbb{R}, \quad f(x_1, \cdots, x_n) := \sum_{i,j=1}^{n} f_{ij} x_i x_j \tag{0.1}$$

be a convex quadratic function. Show that its Legendre transform is also a convex quadratic function

$$g : \mathbb{R}^n \to \mathbb{R}, \quad g(p_1, \cdots, p_n) = \sum_{i,j=1}^{n} g_{ij} p_i p_j \tag{0.2}$$

satisfying

$$g(p(x)) = f(x), \quad f(x(p)) = g(p) \tag{0.3}$$

where $x(p)$ is the unique point maximizing $xp - f(x)$ and $p(x)$ is the unique point maximizing $px - g(p)$.

(5) Suppose we have a system satisfying Hamilton's equations with an autonomous Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$. What happens to the phase curves if we replace H with H^2?

(6) (Arnold page 74) **Optional**. Consider the first digit of the numbers 2^n, $n \in \mathbb{N}$. Does the digit 7 appear in the sequence? Does the digit 8 appear more often than 7? Use Poincaré recurrence somewhere in your solution (you might also need some results from measure theory).

(7) (Arnold page 90). Suppose we have a uniform helical line

$$x = \cos(\phi),\ y = \sin(\phi),\ z = c\phi$$

in which a current is running through it. Consider the motion of a charged particle of mass m in the magnetic field generated by this line. Find one parameter family of symmetries of this system and compute the corresponding first integral.