HOMEWORK 1

(1) (Arnold page 10). Suppose we have a mechanical system describing the motion of three points in $\mathbb{R}^3$ and which is invariant under the group of Galilean transformations (assume the force field is smooth). At the initial moment their velocities in some Galilean coordinate system are equal to zero. Show that the points always remain in the plane which contained them at the initial moment.

(2) (Arnold page 20). Consider a system of mass 1 with one degree of freedom with potential energy:

$$U(x) = \frac{x^4}{4} - x^3 + x^2.$$

Sketch the phase curves of this system.

Definition: A *periodic orbit* of a system with n degrees of freedom is a phase curve $\gamma : [0,T] \longrightarrow \mathbb{R}^{2n}$ satisfying $\gamma(0) = \gamma(T)$. The *period* of γ is T. A *simple periodic orbit* is a periodic orbit γ as above with the property that $\gamma|_{[0,T)}$ is injective.

(3) (Arnold page 20). Let $E : \mathbb{R}^2 \to \mathbb{R}$ be the energy of a conservative system with one degree of freedom and let $a \in \mathbb{R}$. Assume that E is a smooth function. Suppose that $E^{-1}((-\infty, e])$ is compact with area $S(e)$ for each $e \leq a$. If C is a simple periodic orbit with image $E^{-1}(e)$ for some $e < a$, then show that the period of this phase curve is equal to $\frac{dS(y)}{dy}|_{y=e}$.

(4) Let ξ be a unique minimum of the potential function U of a system with one degree of freedom. Suppose that $U''(\xi) \neq 0$. Let $T(e)$ be the period of a simple periodic orbit near ξ of energy e. Compute $\lim_{e \to E(\xi)^+} T(e)$.

(5) Consider a system with two degrees of freedom whose potential energy is given by

$$\frac{1}{2}x_1^2 + \frac{1}{2}\omega^2 x_2^2$$

for some $\omega \in \mathbb{R}$. What are the periods of its simple periodic orbits?

(6) **Optional**: Suppose we have a system with n degrees of freedom whose potential energy is

$$\sum_{i=1}^{n} \omega_i x_i^2$$

for some $\omega_1, \cdots, \omega_n \in \mathbb{R}$. What are the periods of its simple periodic orbits?

(7) Consider a system with potential $-r^{\frac{4}{3}}$ where r is the radial coordinate in $\mathbb{R}^3$. Show that there exist solutions that are not determined uniquely by the initial position and velocity.