# Collet-Eckmann Laminations and Newhouse dynamics

A Dissertation presented

by

**Zhuang Tao**

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

**Doctor of Philosophy**

in

**Mathematics**

Stony Brook University

**August 2022**

**Stony Brook University**

The Graduate School

**Zhuang Tao**

We, the dissertation committee for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation

**Marco Martens**
**Professor, Department of Mathematics**

**Scott Sutherland**
**Professor, Department of Mathematics**

**Mikhail Lyubich**
**Distinguished Professor, Department of Mathematics**

**Christopher Bishop**
**Distinguished Professor, Department of Mathematics**

**Liviana Palmisano**
**Associate Professor, Department of Mathematics, KTH**

This dissertation is accepted by the Graduate School

Celia Marshik

Interim Dean of the Graduate School

Abstract of the Dissertation

**Laminations of chaotic and Newhouse dynamics**

by

**Zhuang Tao**

**Doctor of Philosophy**

in

**Mathematics**

Stony Brook University

**2022**

In one dimensional dynamics, there are quadratic unimodal maps where the orbit of the critical point has a rate of expansion, the so-called "Collet-Eckmann Condition". For maps with the Collet-Eckmann Condition, we cannot find any periodic attractors. In holomorphic dynamics in higher dimensions the situation is completely different. In particular, in the thesis, we study holomorphic systems of arbitrary dimensions. In dissipative family of such systems, we are able to construct a "Collet-Eckmann Lamination" in the parameter space. Each leave of this lamination is of codimension-1 and each map in the leaf has a critical point with expanding directions and its orbits has chaotic properties. The topological classes of the $\omega-$limit set of the critical point are stable along each leaf of the lamination. We also observe the Newhouse Phenomenon in this lamination. In particular, there are maps in the lamination which has a critical point with Collet-Eckmann condition and has coexistence of infinitely many periodic sinks. Moreover, the union of periodic sinks accumulates at the $\omega-$limit set of the critical point. In the parameter space, we can also find leafs in Collet-Eckmann Lamination where a generic point has the Newhouse Phenomenon. The closure of the Newhouse points are the same the the closure of the whole lamination.

To my parents, Liang Tao and Xia Qiao

# Table of Contents

**Acknowledgements**

First and foremost, I want to express my deepest gratitude towards my advisor, Marco Martens, who guided me into the beautiful world of dynamical systems. Marco is not only a great mentor, but also a great friend. I have learned so much from him, not only the math, but also many things in life. During the pandemic, we have so many discussions on math and life in person or via zoom. I got a 'Renormalization' not only in dynamical systems, but also in my life. Without his encouragement, I would have stopped on the half way.

I would like to extend my sincere thanks to Liviana Palmisano for the invaluable help during the program. It has been a pleasant journey during the collaboration, and the results are fruitful. I would like also to thank sincerely for the help during the job application.

I am also grateful to other members in the committee, Scott Sutherland, Christopher Bishop and Mikhail Lyubich for their insightful comments on this dissertation and supports during my graduate study.

There are many other professors at Stony Brook who have helped me throughout my time here. Special thanks to Xiuxiong Chen, for his help during the whole program, especially in the pandemic time. I would like to thank Christopher Bishop, Samuel Grushevsky, Claude LeBrun and Aleksey Zinger for the encouragement on mathematics. I would like to thank Sabyasachi Mukherjee and Zhiqiang Li for leading me into the world of holomorphic dynamics. I am also grateful to Eric Bedford for his help and useful suggestions during my research and job application. I am also thankful to Eric Bedford, Xiuxiong Chen, Mark Andrea De Cataldo, Dzmitry Dudko, Kenji Fukaya, Samuel Grushevsky, Lowell Jones, Ljudmila Kamenova, H. Blaine Lawson, Jr., Claude LeBrun, Mikhail Lyubich, Mark McLean, Mikhail Movshev, Dimitrios Ntalampekos, Olga Plamenevskaya, Christian Schnell, Raanan Schul, Jason Starr, Dennis Sullivan, Leon Takhtajan, Dror Varolin, Oleg Viro, Aleksey Zinger for their illuminating lectures.

I would like give my thank to professor Huijun Fan at Peking University, who led me into the world of geometry and mathematical physics in undergraduate years and also the encouragement during my visiting year in BICMR. Special thanks to professor Weiyue Ding for his encouragement.

I also want to express my thanks to my friends and Ph.D. fellows inside or outside Stony Brook, for there help in math and life. A small part of the full list is: Qianyu Chen, Xujia Chen, Zeyu Cao, Gao Chen, Jingrui Cheng, Paul Frigge, Yuan Gao, Fumi Honda, Siqi He, Xuntao Hu, Yizhe Huang, Shaosai Huang, Jin Huang, Rentai Hou, Yilun Jiang, Wenbo Li, Yuhao Liu, Xiangzi Meng, Santai Qu, Jiaxiang Ren, Yuhan Sun, Hanxi Sun, Jian Wang, Yuanqi Wang, Tutu Wan, Daren Wei, Ruijie Yang, Hang Yuan, Boyu Zhang, Letao Zhang, Mu Zhao, Dingxin Zhang, Zili Zhang, Sheng-Yuan Zhao, Zhilin Zou, Fangyu Zou. I appreciate the help from the department staffs, Lynne Barnett, Christine Gathman, Patrick Tonra, Diane Williams, Charmine Yapchin.

Finally, I would like to thank my parents, Liang Tao and Xia Qiao for their unconditional love and support starting from my date of birth. Their belief in me has kept my spirits and motivation high during this process.

The epitaph on David Hilbert's tombstone in Göttingen consists of the famous lines he spoke at the conclusion of his retirement address to the Society of German Scientists and Physicians on 8 September 1930. As my personal motto, it has been encouraging me through out the struggling days and would carry me to the future. And I would like to end with it:

"We must know, we will know."

# Chapter 1

# Introduction

One of the purposes of science is to understand the law of the world. The main theoretical tools for scientists to do so are through modelings using mathematics. We may build up differential equations for continuous time models or recursive relations for discrete time models, which abstractly, can be written as

$$\frac{dX(t)}{dt} = F(X(t), t) \tag{1.0.1}$$

or

$$Y_{n+1} = G\left(Y_n, n\right) \tag{1.0.2}$$

for continuous time $t$ or discrete time $n$ respectively, where $X(t)$ and $Y_n$ are points in some appropriate spaces with finite or infinite dimensions, $F$ and $G$ are maps with certain regularity.

Now suppose we get the right model for our target object, and suppose we know how to solve our equation for any given initial state, can we predict the motion of the target object in a certain amount of time in practice? For example, in physics, in order to know the orbit of a planet, physicists came up with a system of differential equations using Newton's mechanics to describe the motion of a group of celestial objects interacting with each other gravitationally, which is called the $N-$body problem. For $N = 2$, the two body problem is completely understood. The two planets are either trapped in elliptical orbits that share a focus at the center of mass, or they escape along hyperbolic orbits. They will also be confined to a two dimensional Euclidean plane in the three dimensional Euclidean space due to the conservation of angular momentum.

But when $N \geqslant 3$, things are much more complicated. In the striking work of Henri Poincaré [Poi90], he found that in some cases of the 3-body problem, the trajectory of objects will be extremely sensitivity to initial conditions, which we know today as chaos. In these situations, when the initial condition was perturbed a little bit, the resulting trajectories will change dramatically after a certain time. So in such a system, even the system itself is deterministic, due to the inevitable existence of errors in the observation of initial state, it is impossible to predict the state when time is longer than some finite number. Chaos theory was summarized by Edward Lorenz, who introduced the famous Lorenz system in [Lor63], as: Chaos: When the present determines the future, but the approximate present does not approximately determine the future.

The following old proverb may describe the chaotic system metaphorically:

For want of a nail the shoe was lost.
For want of a shoe the horse was lost.
For want of a horse the rider was lost.
For want of a rider the message was lost.
For want of a message the battle was lost.
For want of a battle the kingdom was lost.
And all for the want of a horseshoe nail.

In the dynamical point of view, we can give an explanation like the following:
A small change of the initial states (”Lack of a nail”) will lead to a dramatic change of the state in the long run (”Losing a battle”). For an introduction to mathematical theory of chaos, we refer to the textbooks written by Devaney [Dev89] and Guckenheimer& Holmes [GH83].

So we now know for chaotic systems, one problem of such system is the unpredictability of long time future in practice as the inevitable existence of errors in the observation of initial state. The other problem for chaotic systems in practice is the inevitable existence of modelling errors. In practise, when people build up a model, it may often come up with using many parameters or using some assumptions and approximations. The errors in the observation of parameters or the error occurs during the assumptions and approximations contribute to the error of the model. So it is also very important to know the robustness of the dynamical features we got for the model in practice, which means these dynamical features are the equivalent for any small perturbations of the model. This leads us to consider the stability of the system itself. There are many versions of definitions of such kind of stability, the strongest version of stability conditions is the structural stability introduced by Andronov and Pontryagin [AP37]. In discrete dynamical systems, this notion can be defined as follows.

Let $M$ be a manifold and $f : M \to M$ be a diffeomorphism on $M$ as the dynamical system. Let $\mathfrak{S}$ be the class of diffeomorphisms of certain regularity, which can be $C^r(r \geqslant 1)$, $C^\infty$ or holomorphic. We say $f$ is $\mathfrak{S}$-structurally stable if for any small enough perturbation $g$ of $f$ in the class $\mathfrak{S}$, there exist a homeomorphism $h$ of $M$, such that $h \circ f = g \circ h$ holds for all points in $M$. From this definition we can see that for any point $x \in M$, the orbit of $x$ under $f$ can be mapped one-to-one to the orbit of $h(x)$ under $g$ through the homeomorphism $h$. In other words, the two systems are the same up to change of coordinates.

There are two aspects of understanding the structurally stable systems, one is to understand the orbital structures of the structurally stable systems, and the other is to understand the density of structurally stable systems among all dynamical systems in a given regularity class, measure-theoretically or topologically. Uniform hyperbolicity is observed to give a way to describe the two aspects. In discrete dynamical systems, uniform hyperbolicity can be defined as follows.

Let a $M$ be a manifold and $f : M \to M$ a $C^1$-diffeomorphism. A compact $f$-invariant subset $\Gamma \subset M$ is uniformly hyperbolic if the restriction of the tangent bundle $TM$ to $\Gamma$ splits into two continuous invariant subbundles:

$$TM|_\Gamma = E^s \oplus E^u \tag{1.0.3}$$

and $E^s$ being uniformly contracted and $E^u$ being uniformly expanded. Given a hyperbolic compact set $\Lambda$, for every $p \in \Lambda$, the sets

$$W^s(p) = \left\{c' \in M : \lim_{n\to+\infty} d\left(f^n(p), f^n\left(p'\right)\right) = 0\right\},$$
$$W^u(p) = \left\{c' \in M : \lim_{n\to-\infty} d\left(f^n(p), f^n\left(p'\right)\right) = 0\right\}$$

are called the stable and unstable manifolds of $p$. By invariant manifold theorems, they are immersed manifolds tangent at $p$ to respectively $E^s(p)$ and $E^u(p)$. There are many studies of structural stability, hyperbolic dynamical systems and their relationships. For a mathematical treatment of these topics, we refer to the textbook written by Shub [Shu87] and also the comprehensive introduction to modern dynamical systems written by Katok and Hasselblatt [KH95]. We also refer to a very good lecture by Berger [Ber18] summarizing recent progress on structural stability. Here we cite one classical result:

**Theorem 1.0.1.** *(Anosov [Ano67], proof by Moser [Mos69]) A uniformly hyperbolic compact set $\Lambda$ for a $C^1$-diffeomorphisms is structurally stable.*

The main conjecture on structural stability is the Palis-Smale stability conjecture [PS70]:

**Palis-Smale stability conjecture.** A $C^k$(or $C^s, s \geqslant k$) diffeomorphism is $C^k$ structurally stable if and only if it satisfies Axiom A and the transversality condition.

We now give the definition of Axiom A and strong transversality condition. Here the Axiom A is related to the hyperbolicity. Indeed, a diffeomorphism $f$ is called Axiom A if the following conditions holds:

(1) The non-wandering set, $\Omega(f)$, is a compact hyperbolic set;

(2) The set of periodic points of $f$ is dense in $\Omega(f)$.

Where the non-wandering set $\Omega(f)$ is defined as

$$\Omega(f) := \Big\{x | \text{ for all open } U \ni x \text{ , there exists } N \text{ such that } f^N(U) \cap U \neq \emptyset \Big\} \tag{1.0.4}$$

And for an Axiom A diffeomorphism $f$, we say it satisfies the transversality condition if the stable and unstable manifolds of any two points in $\Omega(f)$ are in general position. The transversality condition is an important, as it can be shown that, the breaking of transversality condition will make the system away from structural stability.

Here we want to emphasis one special cases, which is the homoclinic orbits. For a hyperbolic fixed point $p$, $q$ is a homoclinic point if $q$ is inside the intersection of the unstable and stable manifold of $p$, in other words, we have

$$\lim_{t \to \pm\infty} f^t(q) = p. \tag{1.0.5}$$

We can classify homoclinic points into two categories by transversality condition: If the stable and unstable manifolds of $p$ are transversal at intersection point $q$, then this intersection point is called transverse homoclinic point. Otherwise we call it homoclinic tangency point. The homoclinic point was firstly introduced by Henri Poincaré [Poi90] during the treatment of 3-body problem. The Horseshoe map, introduced by Stephen Smale [Sma67], plays an important role in understanding the dynamics near a transverse homoclinic point. The Horseshoe map is a a topological model with a very simple symbolic dynamics description and it is the first example of a structurally stable diffeomorphism with an infinite number of periodic points [Sma63]. And using this tool, we have the following theorem [Sma65]:

**Theorem 1.0.2.** *Suppose $x$ is a transversal homoclinic point of $f \in \mathrm{Diff}(M)$. Then there is a Cantor set $\Lambda \subset M, x \in \Lambda$, and $m \in Z^+$such that $f^m(\Lambda) = \Lambda$ and $f^m$ restricted to $\Lambda$ is topologically a shift automorphism.*

The chaotic behavior around such maps are actually theoretically known, and it is an example of the so called "Hyperbolic attractors".

But unfortunately, we don't have such beautiful and simple description for dynamics around homoclinic tangency points. And it is beyond the context of classical uniform hyperbolic systems. We need more math around it. Palis gave a conjecture around it in [Pal00](Conjecture II):

**Palis Conjecture.** In any dimension, the diffeomorphisms exhibiting either a homoclinic tangency or a (finite) cycle of hyperbolic periodic orbits with different stable dimensions (heterodimensional cycles) are $C^r$ dense in the complement of the closure of the hyperbolic ones, $r \geq 1$.

Pujals and Sambarino [PS00] have provided a positive solution to the conjecture for surface diffeomorphisms and r = 1. In the dissertation, we are interested in the holomorphic case. More concretely, we will focus on the bifurcation of dynamical systems with quadratic homoclinic tangency associated to a hyperbolic periodic cycle.

There are many interesting dynamical phenomenons around bifurcation of homoclinic tangencies. Two important properties are of our special interest, the Collet-Eckmann Condition and the Newhouse phenomenon. We start with discussing the Collet-Eckmann Condition first.

For expanding maps on an bounded interval(it is an example of hyperbolic systems), we can find an invariant probability measure which is absolutely continuous with respect to the Lebesgue measure. The expansiveness plays an important role in the proof. But if we are class of unimodal maps (an example of such families are the quadratic family:$f_c(x) = x^2 + c, c \in [-2, \frac{1}{4}]$), because the existence of the critical point, there are no universal expansion. Hence such an invariant measure absolutely continuous with respect to the Lebesgue measure may not exist for all the unimodal maps. Collet and Eckmann [CE83] firstly introduced an weak condition, now called The Collet-Eckmann Condition, on the growth rate of post-critical point to guarantee the existence of invariant probability measures which is absolutely continuous with respect to the Lebesgue measure. Let $c$ be the critical point of a unimodal map $f$, then the Collet-Eckmann Condition can be defined as:

$$|Df^n(f(c))| \geqslant K\lambda^n, \text{ for all } n \geqslant 0, \tag{1.0.6}$$

where $K > 0$ and $\lambda > 1$ are constants. The maps with Collet-Eckmann Condition have shared many fine properties with the classical hyperbolic systems: exponential decay of correlations, the central limit and large deviations theorems, good spectral properties and zeta functions, see [KN92], [You92] for details.

For unimodal $C^1$ maps on an compact interval satisfying the Collet-Eckmann condition together with some other technical conditions, they showed the existence of invariant probability measures which is absolutely continuous with respect to the Lebesgue measure. Many papers later have addressed this question, e.g. Nowicki and van Strien [NS88], Benedicks and Carleson [BC85] and Tsujii [Tsu93]. For interested readers, we refer the classical textbook [MS93] for detailed treatment. One may ask whether the Collet-Eckmann condition a typical condition among certain family of maps. The answer is actually yes.

As having the absolutely continuous invariant measure can be viewed as the "stochastic" property, Lyubich has shown in [Lyu02] that almost every real quadratic map is either regular or stochastic. A quadratic map $P_c : x \mapsto x^2 + c$ is called regular if it has an attracting cycle. In this case, the attracting cycle is unique and attracts almost all orbits [Sin78],[Guc79]. It is called stochastic if it has an absolutely continuous invariant measure. In this case the measure is unique, weakly Bernoulli, and almost all orbits are asymptotically equidistributed with respect to it [Led81], [BL91]. Furthermore, in the works of the Avila and Moreira [AM05],[AM03], the statistical properties of quadratic family or S-unimodal maps are analyzed, they showed a dichotomy of regular and stochastic behaviors of maps, roughly speaking, there exist a full measure subset in stochastic parameters of the family satisfying the Collet-Eckmann condition.

The Collet-Eckmann condition is also considered in holomorphic dynamics, many papers have considered the properties of rational Collet-Eckmann maps over Riemann spheres, e.g., Przytycki [Prz98], Graczyk and Smirnov [GS98], Magnus[Asp04], Astorg, Gauthier, Mihalache and Vigny [Ast+19].

For dynamical systems of dimensions greater than 1, one notable usage of the Collet-Eckmann condition is the the celebrated work of Benedicks and Carleson [BC91] on the the dynamics of the Hénon map. They have shown in the following celebrated theorem for the real Hénon family $T : (x, y) \mapsto (1 + y - ax^2, bx)$:

**Theorem 1.0.3.** *(**Benedicks, Carleson**) Let $W^u$ be the unstable manifold of $T$ at its fixed point in $x, y > 0$. Then for all $c < \log 2$ there is a $b_0 > 0$ such that for all $b \in (0, b_0)$ there is a set $E(b)$ of positive one-dimensional Lebesgue measure such that for all $a \in E(b)$ :*
*(i) There is an open set $U = U(a, b)$ such that for all $z \in U$,*

$$\operatorname{dist}\left(T^\nu(z), \overline{W^u}\right) \to 0 \quad \textit{as } \nu \to \infty$$

*(ii) There is a point $z_0 = z_0(a, b) \in W^u$ such that*
*(a) $\{T^\nu(z_0)\}_{\nu=0}^\infty$ is dense in $W^u$;*
*(b) $\|DT^\nu(z_0)(0, 1)\| \geq e^{c\nu}$.*

In Theorem 1.0.3, property $(ii)(b)$ is also called the Collet-Eckmann condition. There are many great studies following Benedicks and Carleson's methods, e.g., Mora and Viana [MV93], Wang and Young [WY01],[WY08], Viana and Lutstsatto [VL03], Takahasi [Tak11]. And in this dissertation, we will define the Collet-Eckmann condition in similar manner.

Another property which is of our interest is the celebrated Newhouse phenomenon [New74], which is, roughly speaking, the existence of maps with coexisting infinitely many sinks, plays an important role in the theory of non-hyperbolic dynamical systems. Many theorems and conjectures are related to this phenomenon. The classical theory makes inductively use of the idea of persistent tangencies and a deep analysis of intersections of Cantor sets. Then we can prove that for certain class of diffeomorphisms some manifold $M$, there exist a open subset which consist of diffeomorphisms with persistent tangencies and therefore, a generic diffeomorphism in this open subset would carry infinitely many sinks. For a detailed explanation about this topic, we refer the classical textbook [PT93]. Recently in the work of Berger [Ber16], the author introduced the notion of $C^d-$paratangency and parablender to prove that the following theorem:

**Theorem 1.0.4.** *(**Berger**) For all $\infty \geqslant r > d \geqslant 0$ or $\infty > r = d \geqslant 2$, for all $k \geqslant 0$, we have the following:*

*(1) If $M$ is a compact surface, then there exists an open set $\widehat{U}$ in $C^d\left(\mathbb{R}^k, C^r(M,M)\right)$ and a Baire residual set $\mathcal{R}$ in $\widehat{U}$ so that for every $(f_a)_a \in \mathcal{R}$, for every $|a| \leq 1$, the map $f_a$ has infinitely many sinks.*

*(2) If $n \geq 3$, then there exists an open set $\widehat{U}$ in $C^d\left(\mathbb{R}^k, \text{Diff}^{\,r}(M,M)\right)$ and a Baire residual set $\mathcal{R}$ in $\widehat{U}$ so that for every $(f_a)_a \in \mathcal{R}$, for every $|a| \leq 1$, the map $f_a$ has infinitely many sinks.*

For Newhouse Phenomenon in the dynamical systems of several complex variables, it was firstly constructed in the work of Buzzard [Buz97] and Gavosto [Gav98] for polynomial maps with high degree. Recently, Biebler [Bie20] have constructed a residual set in the space of automorphisms of low degree in $\mathbb{C}^3$ which consist of parameters satisfying the Newhouse phenomenon. For polynomial automorphisms of $\mathbb{C}^2$, Dujardin and Lyubich [DL15] have certified a holomorphic version of Palis conjecture [Pal00] by showing that for a family of dissipative polynomial automorphisms of $\mathbb{C}^2$, the set of parameters satisfying either locally weakly $J^*$-stable condition or Newhouse phenomenons is dense in the parameter space.

There are many great articles around all the subjects we have mentioned above. We apologize if there are any missing of references and citations.

## 1.1 Methodology and Statement of main theorems

Recently, Benedicks, Martens and Palmisano in [BMP18] started to study the stability of the Newhouse phenomenon arsing from unfoldings of homoclinic tangencies for dissipative

$C^\infty$ real maps. They are able to prove that there are codimension 2 laminations, named as "Newhouse Laminations",in the parameter space consisting of maps with infinitely many sinks . And the sinks moves smoothly along the leaves of the lamination. And based on their result, Martens, Palmisano and I in [MPT20] have extended the result to polynomial self-maps on $\mathbb{C}^2$. The analysis in the two works are all based on real analysis and there are some obstructions to extend the result into holomorphic settings.

In this dissertation, we are able to extend the results to dynamical systems in several complex variables. Both the parameter space and the phase space will be holomorphic in this dissertation. We follow the same combinatorics aspects of dynamics as [BMP18]and [MPT20], but we reconstruct everything using purely tools from complex analysis. In this way, we are able to give a better geometrical picture and can be applied for future developments.

Besides, following the ideas from Benedicks and Carleson [BC91], we also study the stability of the Collet-Eckmann condition. After showing there is a complex codimension-1 lamination in the parameter space of maps with the Collet-Eckmann condition, we are able to find a dense subset of the lamination of maps with Newhouse phenomenon. This subset can be viewed as the extension of the real Newhouse Laminations constructed in Benedicks, Martens and Palmisano in [BMP18] to the holomorphic setting.

This dissertation is mostly self-contained. After reviewing some basic tools from complex analysis, all the theorems and lemmas are proved in the dissertation.

In the dissertation, we restrict ourselves in family of dynamical systems with unfoldings of strong homoclinic tangency. We have define the notion of strong homoclinic tangency and the unfolding of it in Section 2.4.

Now we give a definition of the Collet-Eckmann condition in our setting. Before give the actual definition, we need to define the meaning of critical point in our setting, which we named as a "quasi-critical point". It is given in section 2.5, see definition 2.5.1. Figure 1.1 gives a model illustration of "quasi-critical point" in the real slice.

Then we can give the definition of Collet-Eckmann condition.

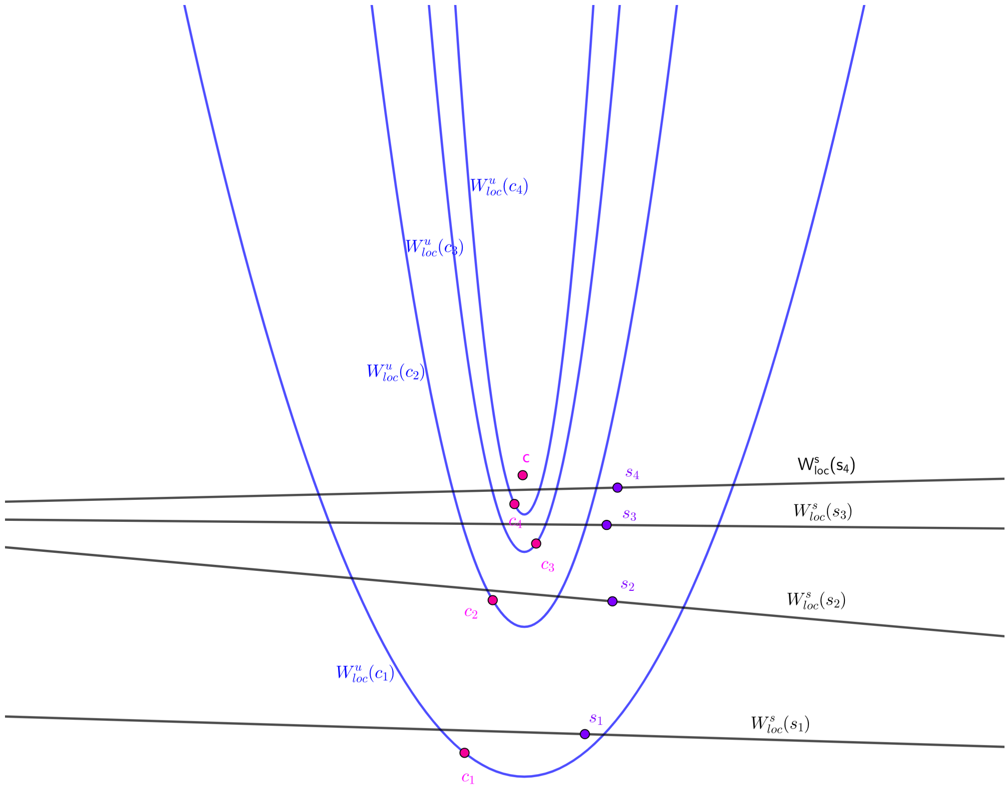


Figure 1.1: Model picture of "quasi-critical point" in the real slice

**Definition 1.1.1.** We say a dynamical system $F : M \longrightarrow M$ satisfies the Collet-Eckmann condition if the following holds:

There exist constants $K, \rho > 1$, a point $c \in M$ which is a quasi-critical point of $F$ and a vector $V$ of unit length in the tangent space of $M$ at $c$, such that the following holds for every positive integer $n$:

$$|DF^n(c)(V)| > K \cdot \rho^n. \tag{1.1.1}$$

Our main results on around the Collet-Eckmann condition is the followings:

**Theorem A.** Let $M$, $\mathcal{P}$ be complex manifolds with $\dim(M) \geq 2$ and $\dim(\mathcal{P}) \geq 3$. Let $F : \mathcal{P} \times M \to M$ be a holomorphic family of unfolding of a map with a strong homoclinic tangency. Then we have the following results:

There exist a codimension 1 lamination in $P$, denoted as $\mathfrak{CE}$, satisfies the following:

(1) All the leafs in the lamination $\mathfrak{CE}$ can be viewed as graphs over a fixed domain with uniform radius.

(2) For any parameter $t$ inside $\mathfrak{CE}$, there exist a quasi-critical point $c(t) \in M$ and a vector $V(t)$ of unit length in the tangent space of $M$ at $c(t)$, such that $F_t$ satisfy the Collet-Eckmann condition at $c(t)$ in the direction $V(t)$ for some $K, \rho > 1$ uniformly, i.e.,

$$|DF_t^n(c(t))(V(t))| > K \cdot \rho^n. \tag{1.1.2}$$

(3) $c(t)$ and $V(t)$ persist along each leave of the lamination holomorphically. Also, the combinatorics of $c(t)$ under the map $F$, which is given by the kneading sequence (see remark 6.2.3), will be constant along the leaves. On the contrary, different leaves will have different kneading sequences.

(4) The closure of $\mathfrak{CE}$ is a lamination. Furthermore, the transversal sections of the lamination $\overline{\mathfrak{CE}}$ are cantor sets, i.e., totally disconnected perfect sets.

(5) The $\omega$-limit set of $c(t)$, $\omega(c(t))$, forms a cantor set. $c(t)$ is a recurrent point, i.e., $c(t) \in \omega(c(t))$, and $c(t)$ is transitive in $\omega(c(t))$.

*Remark* 1.1.1. Let $t$ be a parameter in a leaf $L$ of the lamination $\mathfrak{CE}$, we emphasize that $\omega(c(t))$ is not stable in the parameter space in the following sense. For any complex embedded disc $U$ in the parameter space centering at $t$ with small radius, and transversal to the leaf $L$ at $t$. We may holomorphically extend $c(t)$ to $t' \in U$, but $c(t')$ will no longer satisfies the Collet-Eckmann condition and $\omega(c(t))$ will no longer conjugate to the $\omega(c(t'))$ for any $t' \in U, t' \neq t$. In particular, $\omega(c(t))$ is is not hyperbolic for every $t \in L$.

For the Newhouse Phenomenon, we have the following theorem:

**Theorem B.** With the same assumptions in Theorem A, there exist a set $\mathfrak{NH}$ inside $\mathfrak{CE}$, such that there exists infinitely many sinks for maps with parameter in $\mathfrak{NH}$. And $\mathfrak{NH}$ has following properties:

(1) The closure of $\mathfrak{NH}$ is the same as the closure of $\mathfrak{CE}$, i.e.,

$$\overline{\mathfrak{NH}} = \overline{\mathfrak{CE}} \tag{1.1.3}$$

(2) There exist leaves in $\mathfrak{CE}$, such that the intersection of $\mathfrak{NH}$ with that leaf are dense in the leaf.

(3) Let $\sigma \in \mathfrak{NH}$, then there exists sequence of sinks $\{P_\sigma^{(l)}\}$ with strictly increasing periods, such that the following holds:

$$\omega(c(\sigma)) = \overline{\cup_l O(P_\sigma^{(l)})} \setminus \cup_l O(P_\sigma^{(l)}), \tag{1.1.4}$$

where $O(P_\sigma^{(l)})$ denotes the orbit of the sink.

## 1.2 Further Discussions and Questions

Let $L$ be a leaf in the lamination $\mathfrak{CE}$, for any $t \in \mathfrak{CE}$, by Remark 1.1.1, we know that $\omega(c(t))$ is not stable in the transversal direction of the leaf $L$. But we may investigate the stability behavior of $\omega(c(t))$ when $t$ moves along the leaf $L$. The starting point is the Theorem A(2), $c(t)$ will have the same kneading sequence when $t$ moves along the leaf $L$. And we make a conjecture on this subject as follows:

**Conjecture.** For two parameters $t_{1,2} \in L$, we have that $\omega(c(t_1))$ is conjugated to $\omega(c(t_2))$. The conjecture is not proved in the dissertation. But the reason why we think it is true is the following:

Suppose $t_0$ is a parameter on a leaf $L$ of lamination $\mathfrak{CE}$ where there exist a sink. By section 4, we can find a box in the phase space such that its return map is a Hénon-like map with image completely inside it. Then when we perturbing the $t_0$ along the leaf, the relative position of the box with the image of the same iteration of maps will change continuously to other possibilities. Figure 1.2 shows some of the relative position in the real slice of the phase space. We call it a Hénon renormalization region for cases $(b) - (e)$.

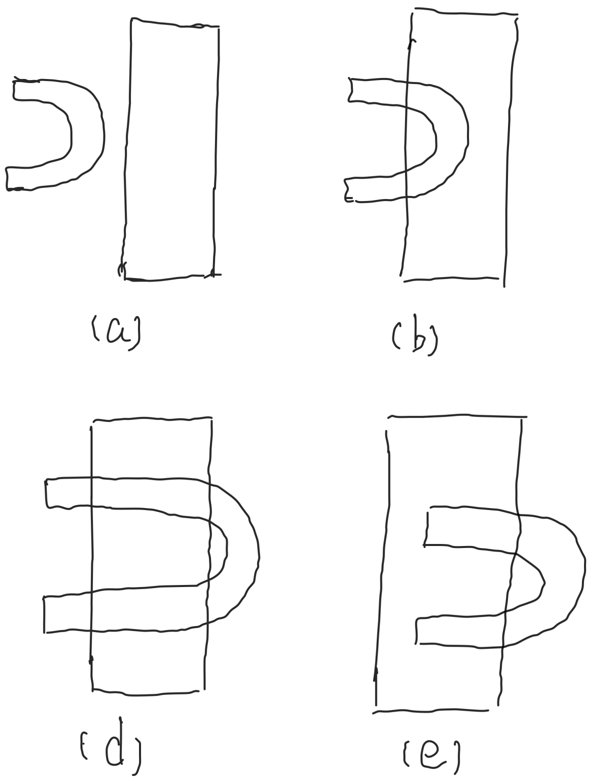


Figure 1.2: Relative position of the box with the image in the real slice. $(a)$ is the case when they do not intersect, $(c)$ is the case of sink, $(b)-(e)$ are the cases when they intersect with each other.

Now we want to see the corresponding regions of the Hénon renormalization region in the parameter space. We specify the parameter space to the one using $(\mu, \lambda, a)$ as coordinates, where $a$ is the unfolding parameter, $\mu$ is the only unstable eigenvalue of the differential of our map at the associated hyperbolic fixed point $p$, $\lambda$ is the unique largest stable eigenvalue of the differential of our map at $p$ (See section 2.4 for details). Then the $a$ axis is transversal to the lamination $\mathfrak{CE}$. And we can actually find 2 strips in the $(\mu, \lambda)-$plane, while the strip, named as Hénon strip, $H$ are made of the parameters such that the box intersect with the image of return map (cases $(b)-(e)$ in Figure 1.2), while a sink strip contained in $H$ consisting of parameters with sinks what got by continuation of the sink at $t$ (case $(c)$ in Figure 1.2). See

Figure 1.3 for the picture in the real slice of $(\mu, \lambda)$ plane. So if parameter $t$ walk through this Hénon strip $H$, a sink will be created and disappeared in the phase space. For a parameter

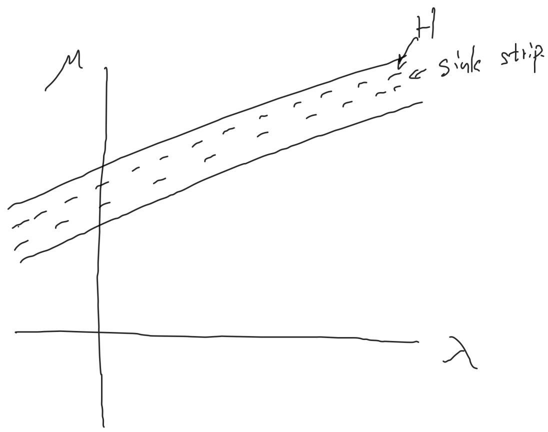


Figure 1.3: Model picture of $H$ and sink strip in the real slice of $(\mu, \lambda)$ plane.

$t_1$ in $\mathfrak{NH}$, there are infinitely many Hénon strips $\{H_i\}$ and corresponding sink strips. Their intersections will contain the point $t_1$. See Figure 1.4 for an illustration in the real slices of $(\mu, \lambda)$ plane. When $t$ is a parameter on leaf $L$ with Newhouse phenomenon, we may have an illustrative picture (figure 1.5). If we are on the leaf with dense subset of Newhouse points as Theorem B, part (2), we can see that there exist an collection of countably many Hénon strips, such that they concentrate on the dense subset of Newhouse points in the leaf. What we can prove is that the complement of all the Hénon strips in the leaf will have positive measure for every leaf in the lamination $\mathfrak{CE}$.

But surprisingly, by the construction of the $c(t)$ for $t \in L$, we can see that $\omega(c(t))$ will never enter any of the Hénon renormalization regions. Thus we can see that all the Hénon strips in the parameter space will not affect the structure of $\omega(c(t))$.

When $t$ moves along the leaf, the Hénon renormalization regions actually will change

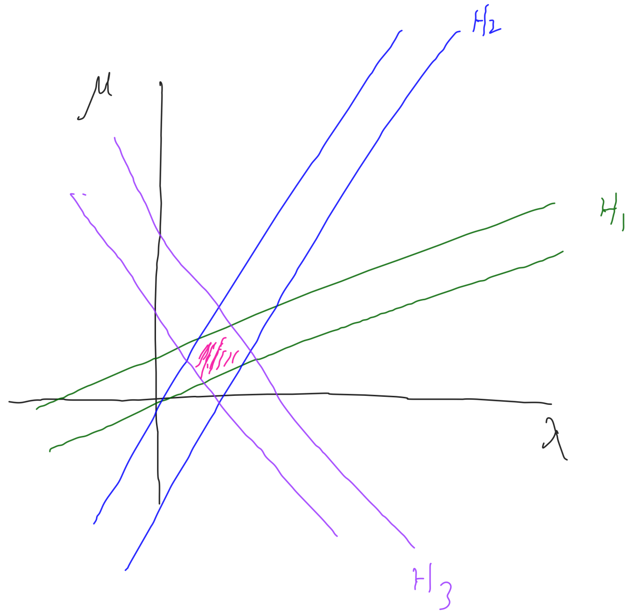


Figure 1.4: Model picture of intersection of Hénon strips at a Newhouse point $t$ in the real slice of $(\mu, \lambda)$ plane.

dramatically as figure 1.2 indicates, the birth and death of sinks can be viewed as a form of topological instability. Furthermore, when $t$ is a Newhouse parameter, by part (3) of Theorem B, we can see that $\omega(c(t))$ and the union of orbits of the sinks have a close relationship: they share the same boundary! So if the conjecture holds, then in the neighborhood of the cantor set $\omega(c(t))$, we can split it into two regions, a "Stablity" region and a "Instablilty" region. The "Stablity" region is $\omega(c(t))$, while the "Instablilty" region consists of unions of all Hénon renormalization regions.

The two ingredients in our discussion, the limit set of a point with Collet-Eckmann condition and the regions creating the Hénon renormalization (and corresponding little Hénon strips in the parameter space), also have appeared in Benedicks-Carleson's work [BC91] on real Hénon maps. Thus we may also ask whether there is a form of stability conditions applies to Benedicks-Carleson's situation. Notice that even though the Hénon renormalization regions will not affect $\omega(c(t))$ in our case, but it will play an role in substantially in Benedicks-Carleson's situation, as the transitivity of the Collet-Eckmann point in the strange attractor.

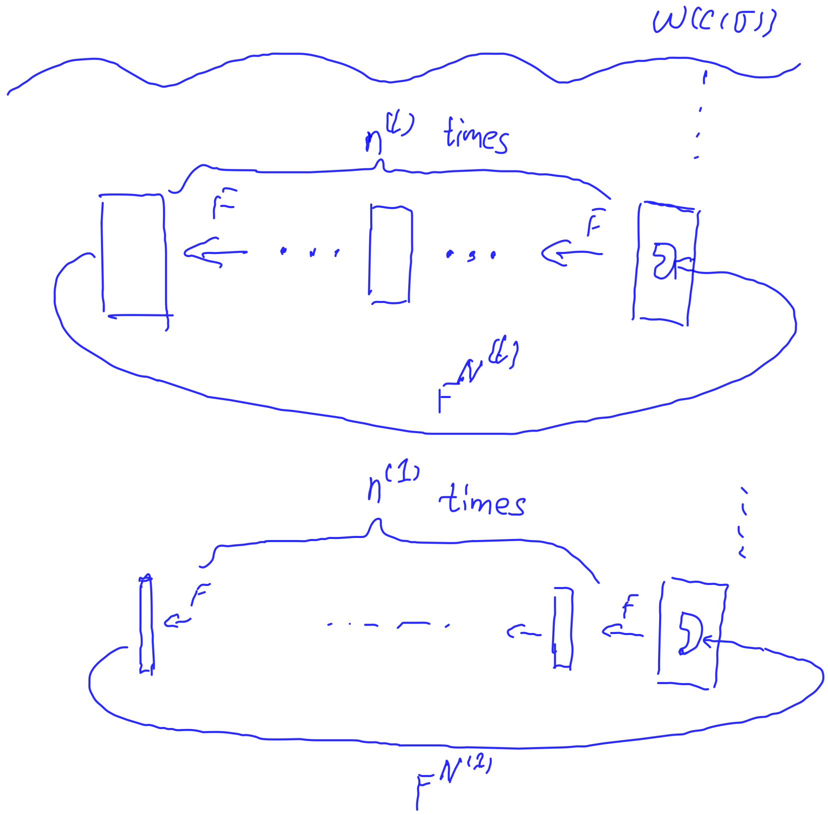


Figure 1.5: Relative position of $\omega(c(t))$ and basins of sinks for a Newhouse parameter $t$

So we need to exclude the potentially affecting Hénon strips in order to get a stability condition, and we expect the set left after these procedures remain positive measure. So we end our discussion with the following question:

**Question.** In theorem 1.0.3, for a real Hénon family $T$, is there an submanifold $P$ in the parameter space such that the Collet-Eckmann point $z_0$ move analytically with the same kneading sequence? If so, is there a subset in $P$ with positive Lebesgue measure (of $P$), such that the resulting strange attractor persist analytically along $P$ with same combinatorics?

# Chapter 2

# Preliminaries

## 2.1 Univalent functions and an Univalency Criteria

A domain in the complex plane $\mathbb{C}$ is an nonempty open connected subset. Let $E$ be a domain in $\mathbb{C}$, a holomorphic function $f : E \longrightarrow \mathbb{C}$ will be called a univalent function if it is injective. The following is a criteria for univalency, the proof can be found in [Pom92].

**Proposition 2.1.1.** *Let $f$ be non-constant and analytic in the domain $H \subset \widehat{\mathbb{C}}$ and let $G$ be the inner domain of the Jordan curve $J$.If*

$$f(z) \to J \quad \text{as} \quad z \to \partial H,$$

*then $f(H) = G$. If furthermore $f$ assumes in $H$ some value in $G$ only once (with multiplicity 1) then $f$ is injective and $H$ is simply connected.*

For univalent map, we have following distortion estimations, we refer to [McM94] and [FM08] for details.

**Theorem 2.1.2.** *Let $D \subset U \subset \mathbb{C}$ be bounded domains with $Mod(D, U) > m > 0$. Let $f : U \to \mathbb{C}$ be a univalent map. Then there is a constant $C(m)$ such that for any $x, y$ and $z$ in $D$, the following holds:*

$$\frac{1}{C(m)} \left| f'(x) \right| \leqslant \frac{|f(y) - f(z)|}{|y - z|} \leqslant C(m) \left| f'(x) \right|.$$

## 2.2 Polynomial-like maps and its perturbations

Following Douady-Hubbard[DH85], we have the definition of polynomial-like maps:

**Definition 2.2.1.** A **Polynomial-like map** of degree $d$ is a triple $(U, U', f)$ where $U$ and $U'$ are open subsets of $\mathbb{C}$ isomorphic to discs, with $U'$ relatively compact in $U$, and $f : U' \longrightarrow U$ a holomorphic mapping, proper of degree $d$. If $d = 2$, we call it a **Quadratic-like map**.

Now we prove a proposition about perturbation of proper maps of a given degree in the complex plane. First we give a technical lemma. Denote $\Delta$ be the unit disc in the complex plane $\mathbb{C}$.

**Lemma 2.2.1.** *(1) Let $d \geqslant 2$ be a positive integer, then there exists positive constants $R_d = \sqrt{\frac{d-1}{d+1}} \in (0,1)$, $\epsilon_d = \frac{2d}{d+1}(\sqrt{\frac{d-1}{d+1}})^{d-1} \in (0,1)$ only depending on $d$ and a increasing analytic function $M_d : [0, \epsilon_d] \to [0, R_d]$ such that the following holds:*
*for any $\epsilon \in (0, \epsilon_d)$, any holomorphic map $f : \Delta \to \Delta$ and any number $u \in \mathbb{C}$ with $|u| = 1$, define $g(z) = uz^d + \epsilon f(z)$ a holomorphic function on $\Delta$, let $\Delta_r$ be the disc centered at origin with radius $r$. Then $g : \Delta_r \to \mathbb{C}$ has $d-1$ critical points counting with multiplicities when $r \in (M_d(\epsilon), R_d)$. Furthermore, we have*

$$M_d(\epsilon) < \epsilon^{\frac{1}{d-1}}. \tag{2.2.1}$$

*(2) When $d = 2$, for any $\epsilon \in (0, \epsilon_2)$ and $\alpha \in (0,1)$, let $t^*(\frac{1-\alpha}{\epsilon}) \in (0,1)$ be the unique solution of the following equation in $(0,1)$:*

$$(1-t^2)^2 = \frac{1-\alpha}{\epsilon} t. \tag{2.2.2}$$

*$t^*(\frac{1-\alpha}{\epsilon})$ is analytic function depending on $\frac{1-\alpha}{\epsilon}$. Then for any $t \in (t^*(\frac{1-\alpha}{\epsilon}), 1)$, we have the following:*
*Let $f : \Delta \to \Delta$ be a holomorphic map with $t < |f(z)| < 1$ for $z \in \Delta$, $u$ a complex number with modulus 1. Consider the function $g(z) = uz^2 + \epsilon f(z)$ on $\Delta$. Then for any $r \in (M_2(\epsilon(1-t^2)), R_d)$, $g : \Delta_r \longrightarrow \mathbb{C}$ has unique critical point. Denote the corresponding*

*critical value as $v(g)$. Then we have the following lower-bound estimation:*

$$|v(g)| > \alpha t\epsilon. \tag{2.2.3}$$

*Proof.* It suffice to prove the lemma for $u = 1$ in both parts.

(1) Since $g'(z) = dz^{d-1} + \epsilon f'(z)$. By Rouché's theorem, $g'(z)$ will have $d-1$ zeros(counting with multiplicities) in the disc $\Delta_r = \{z \big| |z| < r\}$ if the following inequality holds:

$$\epsilon |f'(z)| < d|z|^{d-1} \tag{2.2.4}$$

on the circle $S_r = \{z \big| |z| = r\}$. By Schwarz–Pick theorem, we have

$$|f'(z)| \leqslant \frac{1 - |f(z)|^2}{1 - |z|^2} \leqslant \frac{1}{1 - |z|^2}$$

Thus, inequality (2.2.4) will hold on $S_r$ if we have

$$\frac{\epsilon}{1 - r^2} < dr^{d-1} \tag{2.2.5}$$

Thus we consider the real function $H(r) : (0, 1) \to \mathbb{R}$ defined by

$$H(r) = dr^{d-1} - dr^{d+1}$$

Then we have

$$H'(r) = d(d+1)r^{d-2}\left(\frac{d-1}{d+1} - r^2\right)$$

Thus when $r \in (0, \sqrt{\frac{d-1}{d+1}})$, $H'(r) > 0$, implying $H(r)$ is strictly increasing on $(0, \sqrt{\frac{d-1}{d+1}})$, hence invertible. Thus denote $R_d = \sqrt{\frac{d-1}{d+1}}$, $\epsilon_d = H(\sqrt{\frac{d-1}{d+1}}) = \frac{2d}{d+1}(\sqrt{\frac{d-1}{d+1}})^{d-1}$ and $M_d(\epsilon) = H^{-1}(\epsilon)$ on $[0, \epsilon_d]$. Then when $\epsilon \in (0, \epsilon_d)$, for any $r \in (M_d(\epsilon), R_d)$, inequality (2.2.4) holds, hence $g(z)$ will have $d-1$ critical points counting with multiplicities on $\Delta_r$. For the last inequality, one can check when $r \in (0, \sqrt{\frac{d-1}{d+1}})$, we have

$$H(r) > r^{d-1}.$$

Taking inverse function on the both side, we have

$$r > H^{-1}(r^{d-1}).$$

Thus we have

$$M_d(\epsilon) = H^{-1}(\epsilon) < \epsilon^{\frac{1}{d-1}}.$$

This proves the part (1).

(2) Following the same method of part (1), we have $g'(z) = 2z + \epsilon f'(z)$. By Rouché's theorem, $g'(z)$ will have 1 zero (counting with multiplicities) in the disc $\Delta_r$ if the following inequality holds:

$$\epsilon|f'(z)| < 2r \tag{2.2.6}$$

on the circle $S_r = \{z \big| |z| = r\}$. But now by Schwarz–Pick theorem and the assumption $t < |f(z)| < 1$, we have

$$|f'(z)| \leqslant \frac{1-|f(z)|^2}{1-|z|^2} \leqslant \frac{1-t^2}{1-|z|^2}.$$

Thus inequality (2.2.6) will hold on $S_r$ if we have

$$\frac{\epsilon(1-t^2)}{1-r^2} < 2r. \tag{2.2.7}$$

By previous discussion in part (1), it is equivalent to the following:

$$r > M_2(\epsilon(1-t^2)).$$

Thus for any $r \in (M_2(\epsilon(1-t^2)), R_d)$, $g : \Delta_r \longrightarrow \mathbb{C}$ has unique critical point. Next we consider the estimation on $|v(g)|$. Since $g(z) = z^2 + \epsilon f(z)$, we have

$$|v(g)| \geqslant \epsilon t - r^2$$

for any $r \in (M_2(\epsilon(1-t^2)), R_d)$. Take a sequence $r_i \in (M_2(\epsilon(1-t^2)), R_d)$ with $\lim r_i = M_2(\epsilon(1-t^2))$, passing to the limit and use the inequality (2.2.1), we have

$$\begin{aligned} |v(g)| &\geqslant \epsilon t - (M_2(\epsilon(1-t^2)))^2 \\ &> \epsilon t - (\epsilon(1-t^2))^2 \\ &= \epsilon t[1 - \frac{\epsilon}{t}(1-t^2)^2] \end{aligned}$$

When $t \in (t^*(\frac{1-\alpha}{\epsilon}), 1)$, we have

$$(1-t^2)^2 < \frac{1-\alpha}{\epsilon}t.$$

Combining the two inequalities together, we have

$$\begin{aligned}
|v(g)| &> \epsilon t[1 - \frac{\epsilon}{t}(1-t^2)^2] \\
&> \epsilon t[1 - \frac{\epsilon}{t}\frac{1-\alpha}{\epsilon}t] \\
&= \alpha\epsilon t
\end{aligned}$$

which finish the proof. □

*Remark* 2.2.2. Above lemma focus on the behaviors of critical points and values under small perturbations. For the perturbation of zeros of analytic functions, we refer to [Ros69a] and [Ros69b].

**Proposition 2.2.3.** *Let $U, U' \subset \mathbb{C}$ be two bounded simply connected domains and $f : \bar{U'} \longrightarrow \bar{U}$ be a holomorphic proper surjective map of degree $d$, $d \geqslant 1$. Denote all the critical points of $f$ by $\omega_1, \cdots, \omega_{d-1}$. Suppose there exists a simply connected domain $T \subsetneq U$ such that $T$ contains all the critical value of $f$, we choose a small enough number $\delta$ with $0 < \delta < d(T, \mathbb{C} - U)$. If $d \geqslant 2$, we also require the following:*
*For any $\omega_i$, $T$ contains $D(f(\omega_i), 2\delta)$, the open disc centered at $f(\omega_i)$ with radius $2\delta$. Then their exists an open neighborhood of $\omega_i$, denoted by $D_i$, such that $f(D_i) = D(f(\omega_i), 2\delta)$. We assume that for rach $i \neq j$, either $D_i = D_j$ or $D_i \cap D_j = \emptyset$.*
*Let $g : \bar{U'} \longrightarrow \mathbb{C}$ be a holomorphic map satisfying $|g(z) - f(z)| < \delta$ for all $z \in \bar{U'}$, then the set $T' = g^{-1}(T)$ is a simply connected domain and $g : T' \longrightarrow T$ is a holomorphic proper surjective map of degree $d$. Furthermore, all the critical values of $g$ on $T'$ is contained in $\underset{i}{\cup} D(f(\omega_i), 2\delta)$.*

*Proof.* Consider the straight line homotopy $H$ from $f$ to $g$ on $\bar{U'}$:

$$H : [0,1] \times \bar{U'} \longrightarrow \mathbb{C}$$

$$H(s,z) = (1-s)f(z) + sg(z).$$

Notice that since $|g(z) - f(z)| < \delta$ for all $z \in \bar{U'}$ and $0 < \delta < d(T, \mathbb{C} - U)$, we can deduce that $T$ is strictly inside $H(s, \bar{U'})$ for every $s \in [0,1]$. Thus for any $z \in T$, we have:

$$\text{wind}(g(\partial U'), z) = \text{wind}(f(\partial U'), z) = d$$

where a proper orientation of $\partial U'$ is chosen. Since $T$ is in one connected component of $\mathbb{C} \backslash g(\partial U')$, we conclude that $g$ is a degree $d$ map from $T'$ onto $T$.When $d = 1$, the conclusion follows easily. When $d \geqslant 2$, it suffice to prove that $T'$ contains all the $d-1$ critical points. Now consider $f$ locally around $\omega_i$, $f : D_i \longrightarrow D(f(\omega_i), 2\delta)$. Then we know that $f$ is a degree $d_i$ proper map on $D_i$ for some $d_i \leqslant d$. Let $\phi_i : (D_i, \omega_i) \longrightarrow (\Delta, 0)$ be the uniformization map such that $\phi_i(\omega_i) = 0$, $\xi_i$ be the affine map from $D(f(\omega_i), 2\delta)$ to $\Delta$ such that $\xi_i(f(\omega_i)) = 0$. Then $\widetilde{f_i} = \xi_i \circ f \circ \phi_i^{-1}$ is a degree $d_i$ map from $\Delta$ to itself with only critical point 0. Thus we have $\widetilde{f_i}(z) = u_i z^{d_i}$ where $|u_i| = 1$. Now let $\widetilde{g_i} = \xi_i \circ g \circ \phi_i^{-1}$. Then we have

$$|\widetilde{g_i}(z) - \widetilde{f_i}(z)| < \delta |\xi_i'| = \frac{1}{2}.$$

Since $\frac{1}{2} < \epsilon_d = \frac{2d}{d+1}(\sqrt{\frac{d-1}{d+1}})^{d-1}$, by lemma 2.2.1, we have $\widetilde{g_i}$ has $d_i - 1$ critical points on the disc $\Delta_r$ with $r \in (M_{d_i}(\frac{1}{2}), R_{d_i})$. Then all the critical values are contained in a disc $\Delta_{s(r)}$, where $s(r) = r^{d_i} + \frac{1}{2}$. Choose a sequence $\{r_j\}$ in $(M_{d_i}(\frac{1}{2}), R_{d_i})$ such that $\lim_j r_j = M_{d_i}(\frac{1}{2})$. We conclude that all the critical values are contained in the closed disc $\overline{\Delta_{s(M_{d_i}(\frac{1}{2}))}}$. Since

$$s(M_{d_i}(\frac{1}{2})) = (M_{d_i}(\frac{1}{2}))^{d_i} + \frac{1}{2} < (\frac{1}{2})^{\frac{d_i}{d_i - 1}} + \frac{1}{2} < 1.$$

Thus all the critical values of $\widetilde{g}$ on $\Delta_r$ with $r \in (M_{d_i}(\frac{1}{2}), R_{d_i})$ are in the unit disc $\Delta$. Since $\xi_i$ and $\phi_i$ are biholomorphisms, we know that there are $d_i - 1$ critical values counting with multiplicities in $D(f(\omega_i), 2\delta)$ for the map $g$ corresponding to the local map $f : D_i \longrightarrow D(f(\omega_i), 2\delta)$. Combining all the local information together, we know that $T'$ is homeomorphic to a disc. Thus $T'$ is a simply connected domain and $g : T' \longrightarrow T$ is a holomorphic proper surjective map of degree $d$, and all the critical values of $g$ on $T'$ is contained in $\bigcup_i D(f(\omega_i), 2\delta)$. $\square$

The following proposition is given in Douady-Hubbard ([DH85])(Example 4 in Chapter 1), which can be deduced as a corollary from previous proposition.

**Proposition 2.2.4.** *(Perturbation of Polynomial-like map)([DH85]). Let* $f : U' \longrightarrow U$ *be a polynomial-like map of degree* $d$ *with* $d \geqslant 2$. *Denote its critical points as* $\omega_1, \cdots, \omega_{d-1}$, *counting with multiplicities. Choose* $\delta$ *satisfying* $0 < \delta < \mathrm{d}(U', \mathbb{C} - U)$ *and let* $U_1$ *be the component of* $\{z | \, \mathrm{d}(z, \mathbb{C} - U) > \delta\}$ *containing* $U'$. *Suppose that* $\delta$ *is so small that* $U_1$ *contains all the critical values* $f(\omega_i)$, *where* $1 \leqslant i \leqslant d-1$. *Then if* $g : U' \longrightarrow \mathbb{C}$ *is an holomorphic function such that* $|g(z) - f(z)| < \delta$ *for all* $z \in U'$, *the set* $U_1' = g^{-1}(U_1)$ *is homeomorphic to a disc and* $(U_1, U_1', g)$ *is a polynomial-like map of degree* $d$.

The following estimation is useful, we refer to [McM94] (Lemma 5.5 in [McM94]) for details.

**Lemma 2.2.5.** *Let* $(U, U', f)$ *be a quadratic-like map with critical value lying in a compact set* $K \subset U$, *and let* $K' = f^{-1}(K)$. *Then we have:*

1. $Mod(f^{-1}(A)) = \frac{Mod(A)}{2}$ *for any annulus* $A \subset U$ *enclosing* $K$.
2. $Mod(K', U') \geqslant \frac{Mod(K,U)}{2}$.

## 2.3 Earle-Hamilton holomorphic fixed point theorem

Earle and Hamilton proved a holomorphic fixed point theorem in [EH70].

**Theorem 2.3.1.** *(Earle-Hamilton) Let* $D$ *be a nonempty domain in a complex Banach space* $X$ *and let* $h : D \to D$ *be a bounded holomorphic function. If* $h(D)$ *lies strictly inside* $D$, *then* $h$ *has an unique fixed point in* $D$.

For a mathematical treatment about it we also refer to [Har03].

## 2.4 The definition of strong homoclinic tangency and its unfoldings

### 2.4.1 The definition of strong homoclinic tangency

The definitions of strong homoclinic tangency and the the unfoldings of strong homoclinic tangencies in the real setting appear in [BMP18]. In [**Zhuang**], the definition was extended to the holomorphic case in $\mathbb{C}^2$. In this paper, we consider the strong homoclinic tangency in $\mathbb{C}^m$.

First, following [NPT83], we give the definition of holomorphic quadratic homoclinic tangency.

**Definition 2.4.1.** Let $M$ be an $m$-dimensional complex manifold and $f : M \to M$ a local holomorphic diffeomorphism, $p$ is a rank-one hyperbolic fixed point. Let $q \in W^u(p) \cap W^s(p)$, we say $f$ has a **holomorphic quadratic homoclinic tangency** if the following holds:
$(T1)$ $W^u(p)$ and $W^s(p)$ intersect at $q$ uniquely in the neighborhood of $q$,
$(T2)$ The holomorphic tangent space $T_q^{1,0}(W^u(p))$ is a subspace of $T_q^{1,0}(W^s(p))$,
$(T3)$ Let $i : T_q^{1,0}(W^u(p)) \longrightarrow W^u(p)$ be a holomorphic map such that $i(0) = q$ and $(\mathrm{d}\, i)_0$ is identity matrix; Let $\pi$ be a holomorphic projection (i.e., $\pi^2 = \pi$) of a neighborhood $U$ of $q$ to a complex 1-dimensional submanifold such that $\pi(q) = q$, $(\mathrm{d}\, \pi)_q(T_q^{1,0}(W^s(p))) = 0$ and $\pi(W^s(p) \cap U) = q$; Let $R = \pi(U)$, we call $R$ the **unfolding manifold** associated to $q$. Then $\pi \circ i$ maps 0 to $q$, $\mathrm{d}(\pi \circ i)_0 = 0$. The second holomorphic derivative $\partial^2(\pi \circ i)_0 : T_q^{1,0}(W^u(p)) \otimes T_q^{1,0}(W^u(p)) \longrightarrow T_q^{1,0}(R)$ is a well-defined quadratic map. We define $Q$ to be the composition of $\partial^2(\pi \circ i)_0$ with the canonical isomorphism $T_q^{1,0}(R) \cong T_q^{1,0}(M)/T_q^{1,0}(W^s(p))$. We require

$$Q : T_q^{1,0}(W^u(p)) \otimes T_q^{1,0}(W^u(p)) \longrightarrow T_q^{1,0}(M)/T_q^{1,0}(W^s(p))$$

to be a non-zero quadratic map. Notice the definition of $Q$ is independent of various choices.

*Remark* 2.4.1. Following the definition, there are coordinate systems $(z_1, \cdots, z_m)$ on a neighborhood of $q$($q$ corresponding to origin) such that $W^s(p)$ and $W^u(p)$ locally have the following:

$W^s(p) = \{z_m = 0\}$,

$W^u(p) = \{z_2 = \cdots = z_{m-1} = 0, z_m = sz_1^2\}$ where $s$ is non-zero complex number.

*Remark* 2.4.2. A geometric consequence is the following:

For a small enough neighborhood $W^u_{loc,q}(p)$ of $q$ in $W^u(p)$, $\pi|_{W^u_{loc,q}(p)} : W^u_{loc,q}(p) \longrightarrow R$, i.e., the projection map $\pi$ restricted on $W^u_{loc,q}(p)$, is a branched double-cover onto its image with unique branch point $q$.

Now we will give our definition of strong homoclinic tangency in the holomorphic setting.

**Definition 2.4.2.** Let $M$ be an $m$-dimensional complex manifold and $f : M \to M$ a local holomorphic diffeomorphism satisfying the following conditions:

$(f1)$ $f$ has a rank one saddle point $p \in M$, with only 1 unstable eigenvalue $|\mu| > 1$ and stable eigenvalues $\vec{\lambda} = (\lambda, \lambda_2, \ldots, \lambda_{m-1})$, where $\lambda$ is the unique one with largest modulus, namely

$$|\lambda| > \max_{2 \le i \le m-1} |\lambda_i|,$$

$(f2)$ $|\lambda||\mu|^3 < 1$,

$(f3)$ f has a holomorphic quadratic homoclinic tangency $q_1 \in W^u(p) \cap W^s(p)$, in general position,namely

$$\lim_{n\to\infty} \frac{1}{n} \log d\left(f^n\left(q_1\right), p\right) = \log|\lambda|,$$

$(f4)$ the direction $0 \neq B \in T_{q_1} W^u(p)$ is in general position, namely

$$\lim_{n\to\infty} \frac{1}{n} \log \left|Df^n_{q_1}(B)\right| = \log|\lambda|,$$

$(f5)$ f has a transversal homoclinic intersection, $q_2 \in W^u(p) \cap W^s(p)$ in general position, namely

$$\lim_{n\to\infty} \frac{1}{n} \log d\left(f^n\left(q_2\right), p\right) = \log|\lambda|.$$

A map with these properties is called a map with **strong homoclinic tangency**.

*Remark* 2.4.3. Let $D_p^u(q_2)$ be the closed local unstable manifold isomorphic to a complex disc, such that $q_2$ is on the boundary of it. Then there exists $N \in \mathbb{N}$ such that $q_3 = f^{-N}(q_1)$ lies in the interior of $D_p^u(q_2)$.

### 2.4.2 The unfolding of a map with strong homoclinc tangency

First, let us define the 1−parameter unfolding of a map $f$ with a holomorphic quadratic homoclinic tangency. Let $\mathbb{D}_r \subset \mathbb{C}$ be a disk centered at 0 with radius $r$.

**Definition 2.4.3.** Given a map $f : M \longrightarrow M$ with a holomorphic quadratic homoclinic tangency, then the 1−**parameter unfolding** of $f$ is a holomorphic map $F : \mathbb{D}_r \times M \longrightarrow M$ with the following properties:

$(U1)$ $F_0 = f$, f has a hyperbolic fixed point $p$ and a holomorphic quadratic homoclinic tangency point $q \in W^u(p) \cap W^s(p)$ is associated to $p$,

$(U2)$ there is a holomorphic map $p : \mathbb{D}_r \longrightarrow M$ such that $p(a)$ is a hyperbolic periodic point of $F_a$ and $p(0) = p$. Following the same notation (i.e. $\pi, i, R$) as in (T3) in Definition 2.4.1, we have an univalent map $q : \mathbb{D}_r \longrightarrow M$ such that $q(0) = q$ and for each $a$, $\pi|_{W^u_{loc,q(a)}(p(a))} : W^u_{loc,q(a)}(p(a)) \longrightarrow R$ is a branched double-cover onto its image and $\pi(q(a))$ is the unique branched point. We also have the 1−parameter family of non-zero quadratic form $Q_a$ with $Q_0 = Q$ defined by

$$Q_a : T^{1,0}_{q(a)}(W^u_{F_a}(p(a))) \otimes T^{1,0}_{q(a)}(W^u_{F_a}(p(a))) \longrightarrow T^{1,0}_q(M)/T^{1,0}_q(W^s_f(p)).$$

Thus the composition of $\pi$ and $q$ is an univalent map, $\pi \circ q : \mathbb{D}_r \longrightarrow R$.

*Remark* 2.4.4. Using the notation in Remark 2.4.1 and above Definition, the 1−parameter unfolding $F$ of $f$, $(z_1, \cdots, z_m)$ on a neighborhood of $q(0)$ ( $q(0)$ corresponding to origin) such that $W^s(p(0))$ and $W^u(p(0))$ locally have the following:

$W^s(p(a)) = \{z_m = 0\}$,

$W^u(p(a)) = \{z_2 = \cdots = z_{m-1} = 0, z_m = sz_1^2 + a\}$ where $s$ is non-zero complex number.

Then we consider the concept of unfoldings of maps with strong homoclinic tangency of certain type. This definition is stronger than the previous notions in ([BMP18]) and ([**Zhuang**]), but it is also a natural definition.

**Definition 2.4.4.** Given a map $f$ with a strong homoclinic tangency with saddle point $p$, $\mathcal{P}$ a holomorphic manifold with complex dimension $k$ greater or equal to 3 with a base point $O$, we consider a holomorphic family $F : \mathcal{P} \times M \to M$ through $f$ with the following properties:
$(F1)$ $F_O = f$;
$(F2)$ A holomorphic function $\hat{p} : P \longrightarrow M$, with $\hat{p}(O) = p$, such that $F_b$ has a rank-one saddle point $\hat{p}(b)$. The saddle point $\hat{p}(b)$ of $F_b$ has unstable eigenvalue $\mu(b)$ and stable eigenvalues $\vec{\lambda}(b) = (\lambda(b), \lambda_2(b), \ldots, \lambda_{m-1}(b))$ , where $\lambda(b)$ is the unique one with largest modulus, namely

$$|\lambda(b)| > \max_{2\leq i\leq m-1} |\lambda_i(b)| ;$$

Moreover, $\mu(b), \vec{\lambda}(b)$ are holomorphic mapping of $b$.
$(F3)$ There exist positive integers $k_1, k_2$ such that the following holds:
Up to biholomorphism, $P$ can be written as $T_{t_1}((\mu_{min})^{\frac{1}{k_1}}, (\mu_{max})^{\frac{1}{k_1}}) \times T_{t_2}((\lambda_{min})^{\frac{1}{k_2}}, (\lambda_{max})^{\frac{1}{k_2}}) \times \mathbb{D}_a(\epsilon) \times U$, where $1 < \mu_{min} < \mu_{max}$, $0 < \lambda_{min} < \lambda_{max} < 1$, $\epsilon > 0$, and $T_{t_1}((\mu_{min})^{\frac{1}{k_1}}, (\mu_{max})^{\frac{1}{k_1}}) = \{t_1 \in \mathbb{C} | (\mu_{min})^{\frac{1}{k_1}} < |t_1| < (\mu_{max})^{\frac{1}{k_1}}\}$, $T_{t_2}((\lambda_{min})^{\frac{1}{k_2}}, (\lambda_{max})^{\frac{1}{k_2}}) = \{t_2 \in \mathbb{C} | (\lambda_{min})^{\frac{1}{k_2}} < |t_2| < (\lambda_{max})^{\frac{1}{k_2}}\}$, $\mathbb{D}_a(\epsilon) = \{a \in \mathbb{C} | |a| < \epsilon\}$, $U$ is a bounded domain in $\mathbb{C}^{k-3}$. Furthermore, for any $(t_1, t_2, a, \tau) \in P$, we assume

$$\mu(t_1, t_2, a, \tau) = (t_1)^{k_1} \tag{2.4.1}$$

and

$$\lambda(t_1, t_2, a, \tau) = (t_2)^{k_2}. \tag{2.4.2}$$

$(F4)$ For $\mu_{max}, \lambda_{max}$, we assume

$$\lambda_{max}(\mu_{max})^3 < 1; \tag{2.4.3}$$

$(F5)$ for each given $(\mu, \lambda, \tau) \in T_\mu(\mu_{min}, \mu_{max}) \times T_\lambda(\lambda_{min}, \lambda_{max}) \times U$, $F_{(\mu,\lambda,0,\tau)}$ is a map with strong homoclinic tangency, and $F_{(\mu,\lambda,a,\tau)}$ is $1-$parameter unfolding of $F_{(\mu,\lambda,0,\tau)}$ for $a \in \mathbb{D}_a(\epsilon)$.

A family of maps $F$ with these properties is called an **unfolding of map with strong homoclinic tangency of type** $(k_1, k_2)$.

*Remark* 2.4.5. Throughout the rest of the paper, we will only consider the case for $k_1 = k_2 = 1$ for simplicity of discussion. Indeed, all the results can be extended to any $(k_1, k_2)$ pair by simply rewrite every $\mu, \lambda$ by $t_1^{k_1}, t_2^{k_2}$ respectively. So in this manner, we will set our parameter space with local coordinates $(\mu, \lambda, a, \tau)$.

Furthermore, when we are not in the process with the analysis of parameter dependence, for the simplicity of writing, we will also use the notation $(t, a)$ where $a$ is the unfolding parameter and $t$ stands for the rest of parameters,i.e.,$t = (\mu, \lambda, \tau)$. Thus we will focus on the family $F_{t,a}$ witch is the unfolding of map with strong homoclinic tangency.

*Remark* 2.4.6. The condition $\lambda_{\max}\mu_{\max}^3 < 1$, see $(F4)$, allows us to choose $\theta \in \left(0, \frac{1}{2}\right)$ such that

$$1 < \lambda_{\min}^{2\theta}\mu_{\min}^3 \text{ and } \lambda_{\max}^{\theta}\mu_{\max} < 1,$$

We can choose any $\theta$ satisfying

$$0 < \theta_0 = \frac{\log \mu_{\max}}{\log \frac{1}{\lambda_{\max}}} < \theta < \frac{3}{2}\frac{\log \mu_{\min}}{\log \frac{1}{\lambda_{\min}}} = \theta_1 < \frac{1}{2}. \tag{2.4.4}$$

**Example 1.** Now we give a degree 4 family of polynomial endomorphisms which are unfolding of map with strong homoclinic tangency in a domain of $\mathbb{C}^2$.

For $m = 2$, let $x_1, x_2$ be 2 nonzero different complex numbers with $|x_1| < |x_2|$, denote $f_\mu(x) := -\frac{\mu}{x_1^2 x_2} x(x - x_1)^2(x - x_2)$ be a polynomial of degree 4, we may consider the following generalized Hénon-like map $H_{\mu,\epsilon,a}$ on $\mathbb{C}^2$:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} f_\mu(x) - \epsilon y + a \\ x \end{pmatrix} \tag{2.4.5}$$

where we assume $|\mu| > 1$. The map $H_{\mu,\epsilon,a}$ is non-invertible when $\epsilon = 0$, invertible when $\epsilon \neq 0$, and its inverse map are

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ \frac{x - a - f_\mu(x)}{\epsilon} \end{pmatrix}. \tag{2.4.6}$$

When $\epsilon, a = 0$, $O = (0,0)$ is a fixed point of $H_{\mu,0,0}$, and we can take $W^s_{loc}(O)$ to be the $y-axis$ and $W^u_{loc}(O)$ to be the graph $\{(f_\mu(x), x) || x| \leqslant 2|x_2|\}$. We have a homoclinic tangency at $(0, x_1)$ and a transversal homoclinic intersection at $(0, x_2)$.

Now we perturb $\epsilon, a$ in a small disc around 0, which means chooseing a small disc $\mathbb{D}_\epsilon(\rho)$ around 0. Let $\mu$ be inside a round annuli $T(\mu_1, \mu_2)$ with $\rho\mu_2^3 < 1$. So we consider triple $(\mu, \epsilon, a)$ in $T(\mu_1, \mu_2) \times \mathbb{D}_\epsilon(\rho) \times \mathbb{D}_a(\rho)$.

Denote the fixed point continued from $O$ to be $P_0(\mu, \epsilon, a) = (p(\mu, \epsilon, a), p(\mu, \epsilon, a))$. Then we have $p(\mu, \epsilon, 0) \equiv 0$ and it is the solution of the equation

$$f_\mu(x) - \epsilon x + a = x. \tag{2.4.7}$$

Thus we have the following asymptotic expansion when $\epsilon, a$ small enough:

$$p(\mu, \epsilon, a) = a((-\frac{1}{\mu - 1}) + (-\frac{1}{(\mu-1)^2})\epsilon + \frac{x_1 + 2x_2}{2x_1x_2}\frac{\mu}{(\mu-1)^3}a + \text{higher order terms}). \tag{2.4.8}$$

Let $t_{1,2}(\mu, \epsilon, a)$ be the 2 eigenvalues of $DH_{\mu,\epsilon,a}(p(\mu, \epsilon, a))$, with

$$t_1(\mu, 0, 0) = \mu, t_2(\mu, 0, 0) = 0.$$

We have that $t_{1,2}$ satisfies the following equation:

$$t^2 - f_\mu^{'}(p(\mu, \epsilon, a))t + \epsilon = 0. \tag{2.4.9}$$

Then we have the asymptotic expansion of $t_1$ and $t_2$:

$$t_1 = \mu + (-\frac{1}{\mu})\epsilon + (\frac{2\mu}{\mu-1}\frac{x_1+2x_2}{x_1x_2})a + h.o.t; \tag{2.4.10}$$

$$t_2 = 0 + (\frac{1}{\mu})\epsilon + 0a + h.o.t. \tag{2.4.11}$$

Now consider the holomorphic map $h : (\mu, \epsilon, a) \longrightarrow (t_1, t_2, a)$, and use $t_1t_2 \equiv \epsilon$, we know the differential of $h$ at $(\mu, 0, 0)$ is

$$Dh((\mu, 0, 0)) = \begin{pmatrix} 1 & -\frac{1}{\mu} & \frac{2\mu}{\mu-1}\cdot\frac{x_1+2x_2}{x_1x_2} \\ 0 & \frac{1}{\mu} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.4.12}$$

Then the Jacobian is

$$Jac_h((\mu,0,0)) = det(Dh((\mu,0,0))) = \frac{1}{\mu} \neq 0. \tag{2.4.13}$$

Thus $h$ is local diffeomorphisms in the neighborhood of $(\mu,0,0)$, we may choose some $\rho$ such that $h$ is local diffeomorphism in $T(\mu_1,\mu_2)\times\mathbb{D}_\epsilon(\rho)\times\mathbb{D}_a(\rho)$.

Now consider the first component of the map $h$ and denote it by $S_{\epsilon,a}$, i.e , $S_{\epsilon,a}(\mu)=t_1$. Since we have $h(\mu,0,0)=(\mu,0,0)$, we know $S_{0,0}(\mu)=\mu$. When $\rho$ small enough, by, formula (2.4.10), we can find a small constant $0 < s < \frac{\mu_2-\mu_1}{5}$ such that

$$|S_{\epsilon,a}(\mu) - S_{0,0}(\mu)| < s \tag{2.4.14}$$

for any $(\mu,\epsilon,a)\in T(\mu_1,\mu_2)\times\mathbb{D}_\epsilon(\rho)\times\mathbb{D}_a(\rho)$. Then by proposition 2.2.3, we know $S_{\epsilon,a}$ is a diffeomorphism from $(S_{\epsilon,a})^{-1}(T(\mu_1+s,\mu_2-s))$ onto $T(\mu_1+s,\mu_2-s)$. Denote $V(\rho) := \bigcup\limits_{|\epsilon|,|a|<\rho}(S_{\epsilon,a})^{-1}(T(\mu_1+s,\mu_2-s))\times\{\epsilon\}\times\{a\}\subset T(\mu_1,\mu_2)\times\mathbb{D}_\epsilon(\rho)\times\mathbb{D}_a(\rho)$. And we also have

$$|(S_{\epsilon,a})^{-1}(t_1)-t_1| = |S_{0,0}\circ(S_{\epsilon,a})^{-1}(t_1) - S_{\epsilon,a}\circ(S_{\epsilon,a})^{-1}t_1| < s \tag{2.4.15}$$

for $t_1\in T(\mu_1-s,\mu_2+s)$. Thus we have

$$T(\mu_1+2s,\mu_2-2s)\times\mathbb{D}_\epsilon(\rho)\times\mathbb{D}_a(\rho)\subset V(\rho)\subset T(\mu_1,\mu_2)\times\mathbb{D}_\epsilon(\rho)\times\mathbb{D}_a(\rho). \tag{2.4.16}$$

Now let $\rho_1 = \frac{\rho}{\mu_2-s}$, we know $h^{-1}$ defined by

$$h^{-1}(t_1,t_2,a) = ((S_{t_1t_2,a})^{-1}(t_1), t_1t_2, a) \tag{2.4.17}$$

is a well-defined injective holomorphic map from $T(\mu_1+s,\mu_2-s)\times\mathbb{D}_{t_2}(\rho_1)\times\mathbb{D}_a(\rho)$ onto $V(\rho)$. And we have

$$\text{Jac}\left(h^{-1}\right) = \det\begin{pmatrix} \frac{\partial S^{-1}}{\partial t_1}+\frac{\partial S^{-1}}{\partial\varepsilon}t_2 & \frac{\partial S^{-1}}{\partial\varepsilon}t_1 & \frac{\partial S^{-1}}{\partial a} \\ t_2 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{\partial S^{-1}}{\partial t_1}t_1 \neq 0. \tag{2.4.18}$$

Thus $V(\rho)$ is biholomorphic to $T(\mu_1+s,\mu_2-s)\times\mathbb{D}_{t_2}(\rho_1)\times\mathbb{D}_a(\rho)$ via map $h$.

Now for $(t_1, t_2, a) \in T(\mu_1 + s, \mu_2 - s) \times \mathbb{D}_{t_2}(\rho_1) \times \mathbb{D}_a(\rho)$. First denote $P(t_1, t_2, a)$ to be $P_0(h^{-1}(t_1, t_2, a))$. And in coordinate, we denote

$$P(t_1, t_2, a) = (p(t_1, t_2, a), p(t_1, t_2, a)).$$

Then denote the local stable and unstable manifold of $P(t_1, t_2, a)$ by $W^{s,u}_{loc}(t_1, t_2, a)$. Then we know $W^s_{loc}(t_1, t_2, a)$ is a graph over $D_y(0, 2|x_2|)$ in $y$-axis.

Denote the foliation $\mathfrak{W}^s(t_1, t_2, a)$ to be the foliation generated by translating $W^s_{loc}(t_1, t_2, a)$ in $x$-direction, and denote $\pi^s(t_1, t_2, a)$ to be the projection map along the foliation into the horizontal plane $\{y = p(t_1, t_2, a)\}$. When we consider $(t_1, t_2, a) \in T(\mu_1 + s, \mu_2 - s) \times \mathbb{D}_{t_2}(\rho_1) \times \mathbb{D}_a(\rho)$. Let $U_{t_1,t_2,a}(r) = \{(x, y) || x - p(t_1, t_2, a)| < r, y = p(t_1, t_2, a)\}$.

First we consider the case when $t_2 = a = 0$. Then under this case, $\pi^s(t_1, t_2, a) = \pi_x$, where $\pi_x$ is the projection onto $x$-coordinate. And $U_{t_1,0,0}(r)$ will be the 1-disc $\{(x, 0)||x| < r\}$ in the $x$-coordinate. Now consider $(\pi_x)^{-1}(U_{t_1,0,0}(r)) \cap W^u_{loc}(t_1, 0, 0)$, it will have 3 connected component when $r$ small enough since $(\pi_x)^{-1}(0) \cap W^u_{loc}(t_1, 0, 0) = \{O, (0, x_1), (0, x_2)\}$. Let $W^u(r)$ be the connected component of $(\pi_x)^{-1}(U_{t_1,0,0}(r)) \cap W^u_{loc}(t_1, 0, 0)$ containing $(0, x_1)$. Then there exist a $r_0 > 0$ such that $\pi^s(t_1, 0, 0)$ is a degree 2 branched cover from $W^u(r_0)$ onto $U(r_0)$ with critical point $(0, x_1)$ and critical value 0.

By proposition 2.2.4, when $\epsilon, a$ small enough, there exists $r_1 < r_0$ such that $(\pi^s(t_1, t_2, a))^{-1}(U_{t_1,t_2,a}(r_1)) \cap W^u_{loc}(t_1, t_2, a)$ have 3 connected component. The projection $\pi^s(t_1, t_2, a)$ on each component have degree $2, 1, 1$ separately. Now let $W^u_{t_1,t_2,a}(r)$ be the connected component such that it is an degree 2 branched covering onto $U_{t_1,t_2,a}(r_1)$ via projection $\pi^s(t_1, t_2, a)$. Let the $y$-coordinate of the critical value of this map be $v(t_1, t_2, a)$.

Now we want to solve equation $v(t_1, t_2, a) = p(t_1, t_2, a)$. Notice that $v(t_1, 0, a) = a$ and $v(t_1, 0, 0) = p(t_1, 0, 0) = 0$. We have

$$\frac{\partial v}{\partial a}(t_1, 0, 0) = 1, \tag{2.4.19}$$

and

$$\frac{\partial v}{\partial t_1}(t_1, 0, 0) = \frac{\partial p}{\partial t_1}(t_1, 0, 0) = 0. \tag{2.4.20}$$

By formula (2.4.8) and (2.4.11), we know

$$\frac{\partial p}{\partial a}(t_1,0,0) = -\frac{1}{t_1-1}, \frac{\partial t_2}{\partial a}(t_1,0,0) = 0. \tag{2.4.21}$$

Thus we have

$$\frac{\partial(v(t_1,t_2,a)-p(t_1,t_2,a))}{\partial a}(t_1,0,0) = 1 + \frac{1}{t_1-1} = \frac{t_1}{t_1-1} \neq 0. \tag{2.4.22}$$

Then by implicit function theorem, there exist $\rho_0 < \rho_1$ such that there exist holomorphic map $g : T(\mu_1+s, \mu_2-s) \times \mathbb{D}_{t_2}(\rho_0) \longrightarrow \mathbb{D}_a(\rho)$ with $g(t_1,0) = 0$ such that

$$v(t_1,t_2,g(t_1,t_2)) = p(t_1,t_2,g(t_1,t_2)). \tag{2.4.23}$$

Now consider $\widehat{h} : (t_1,t_2,a) \longrightarrow (t_1,t_2,a')$, where $a' := a - g(t_1,t_2)$. Then $\widehat{h}$ is invertible by $\widehat{h}^{-1}(t_1,t_2,a') = (t_1,t_2,a'+g(t_1,t_2))$.

## 2.5 The normalization of unfoldings

In this section we normalize previous definitions by change of coordinates and rescaling without loss of generality. Let $F_{t,a} : M \longrightarrow M$ be an unfolding of strong homoclinic tangency. Let $p$ be a hyperbolic fixed point of $F_{0,0}$. There exist a coordinate system $(\vec{x}, y) = (x_1, \cdots, x_{m-1}, y)$ in a small enough neighborhood of $p$ satisfying the following:

(1) $p = (\vec{0}, 0)$. $F_{t,a}$ is hyperbolic on $D = \{(\vec{x}, y) \mid |x_i| \leqslant L, |y| \leqslant L\}$ with $p$ the hyperbolic fixed point. Denote $D_2 = \{(\vec{x}, y) \mid |x_i| \leqslant 2, |y| \leqslant 2\}$, then $L > 2$ is a constant chosen to satisfy $F_{t,a}(D_2) \subseteq D$ and $F_{t,a}^{-1}(D_2) \subseteq D$ for every $(t,a)$. Furthermore by the property of local diffeomorphism, we may assume that there exist $0 < s < 1$ such that $||DF(v)|| > s|v|$ for every point $p \in M$, and nonzero vector $v$ the tangent space of $p$.

(2) $W^u_{loc}(p)$ becomes a disc in y-axis, i.e. $W^u_{loc}(p) = D_y = \{(\vec{0}, y) \mid |y| \leqslant L\}$.

(3) $W^s_{loc}(p)$ becomes a poly-disc in $\vec{x}$-axis, i.e., $W^s_{loc}(p) = D_{\vec{x}} = \{(\vec{x}, 0) \mid |x_i| \leqslant L, 1 \leqslant i \leqslant m-1\}$.

(4) $F_{t,0}$ has homoclinic tangency points at $q_1(t) = (1, x_2(t), \cdots, x_{m-1}(t), 0)$ and $q_3(t) =$

$F^{-N}(q_1(t)) = (0, \cdots, 0, z_3)$, where $x_i(t)$ is holomorphic functions of t and $\|x_i(t)\| \leqslant \frac{1}{2}$, $N$ is a positive constant integer and $|z_3| \in (\frac{1}{\mu_{max}}, 1)$.

The tangent space of $W^u(p)$ at $q_1(t)$ is contained in $T_{q_1(t)}W^s(p)$. Then the unfolding manifolds $R_t$ now become vertical: $R(t) = \{(1, x_2(t), \cdots, x_{m-1}(t), y) \mid |y| \leqslant L\}$, we may identify each $R(t)$ with the $y$-axis. Now denote $\pi_y$ as the projection map onto the $y$-coordinate, then $\pi_y|_{W^u_{loc,q_1(t)}(p)} : W^u_{loc,q_1(t)}(p) \longrightarrow D_y$ is a double branched cover with unique branched point $q_1(t)$. Thus we have the following lemma:

**Lemma 2.5.1.** $(D_y, W^u_{loc,q_3(t)}(p), \pi_y F^N_{t,0})$ *is a quadratic-like map in a neighborhood of* $q_3(t)$ *restricted in the* $y-$*axis with unique critical point* $q_3(t)$ *and corresponding critical value* $0$.

(5) $\pi_y \circ h_t$ is identity map, i.e., $\pi_y q_1(t, a) = a$.

In other words, $(D_y, W^u_{loc,q_3(t,a)}(p), \pi_y F^N_{t,a})$ is a quadratic-like map with unique critical point denoted by $q_3(t, a) = F^{-N}_{t,a}(q_1(t, a))$ and corresponding critical value $a$.

Using Taylor expansion, we may assume

$$DF^N|_{q_1(t,a)+(\Delta x, \Delta y)} = \begin{pmatrix} A & B \\ C & D\Delta x + 2Q\Delta y \end{pmatrix} \tag{2.5.1}$$

where $A, B, C, D, Q$ are bounded matrices holomorphic over coordinates and parameters, $(\Delta x, \Delta y)$ are the coordinates of the point with centered at $q_1(t, a)$, since we have a quadratic homoclinic tangency, we also have $|Q| > Q_0 > 0$. Since $F^N$ is diffeomorphism, we know that $DF^N$ has nonzero determinant, so $C(0, 0)$ is a nonzero vector and we may assume $C$ nonzero for the whole neighborhood of $q_1(t, a)$.

(6) There is a homoclinic intersection point $q_2(t) = (0, z_2)$ such that $W^s(p)$ intersect with $W^u_{loc}(p)$ at $q_2(t)$ transversely and $W^s_{loc}(q_2(t)) = \{(\vec{x}, z_2) \mid |x_i| \leqslant L, 1 \leqslant i \leqslant m-1\}$, where $|z_2| = 2$. Furthermore, there exist a constant integer $M > 0$, such that $F^M$ will send a neighborhood of $q_2$ to a neighborhood of $q_3$ for every parameter in the parameter space. And there exist two constants $\phi_{1,2} > 0$, such that the tangent cones $\{(\vec{v}_x, v_y) || \vec{v}_x| < \phi_1 |v_y|\}$ with base point $p_1$ in the neighborhood of $q_2$ will be mapped into the tangent cone $\{(\vec{v}_x, v_y) || \vec{v}_x| <$

$\phi_2|v_y|\}$ with base point $F^M(p_1)$.

(7) Moreover, the restriction of the mapping to each axis of the coordinate system is linearized, see for example [Mil06].

On the the unstable invariant manifolds, we have

$$F_{\mu,\lambda,a,\tau}(\vec{0},y) = (\vec{0},\mu y), \tag{2.5.2}$$

and on each $x_i$-axis, we have

$$F_{\mu,\lambda,a,\tau}(0,\cdots,x_i,\cdots,0) = (0,\cdots,\lambda_i x_i,\cdots,0), 1 \leqslant i \leqslant m-1 \tag{2.5.3}$$

where $\lambda_1 = \lambda$.

Besides, at the origin, we have

$$DF_{t,a}(\vec{0},0) = \begin{pmatrix} \Lambda(t,a) & 0 \\ 0 & \mu \end{pmatrix},$$

where $\Lambda(t,a)$ is the diagonal $(m-1)\times(m-1)$ matrix with diagonal terms $(\lambda_1,\cdots,\lambda_{m-1})$. Thus, for $(x,y)$ in D we have the following estimate

$$F(\vec{x},y) = (\Lambda\vec{x} + P_s(\vec{x},y), \mu y + P_u(\vec{x},y)), \tag{2.5.4}$$

where $P_u$ is holomorphic functions satisfying $P_u(\vec{x},0)=0$ and $P_u(\vec{0},y)=0$, $P_s=(P_{s,1},\cdots,P_{s,m-1})^T$ is holomorphic mapping satisfying the following:

$$P_{s,i}(0,\cdots,x_j,\cdots,0) = 0, 1 \leqslant i,j \leqslant m-1.$$

$$P_{s,i}(0,\cdots,0,y) = 0$$

Their derivatives at the origin are zero. From above condition, and by using Taylor expansion around origin, we have following estimates:

**Lemma 2.5.2.**

$$P_u(\vec{x},y) = O(\sum_{i=1}^{m-1} x_i y),$$

$$P_{s,i}(\vec{x},y) = O(\sum_{1\leqslant l<j\leqslant m-1} x_l x_j + \sum_{i=1}^{m-1} x_i y).$$

(8) The parameter space are chosen to be $P = T_\mu(\mu_{min}, \mu_{max}) \times T_\lambda(\lambda_{min}, \lambda_{max}) \times \mathbb{D}_a(2) \times U$.

**Definition 2.5.1.** For a dynamical system $F : M \longrightarrow M$, we call a point $c$ a quasi-critical point associated to the hyperbolic point $p$,if there are sequences of points $\{c_i\}$ and $\{s_i\}$, such that the following holds:

(1)

$$\lim_i c_i = c, \lim_i s_i = c. \tag{2.5.5}$$

(2) The local stable manifold of $s_i$ and $c$ are graph over the $x$-axis, i.e., $\pi_x$ are biholomorphism from $W^u_{loc}(s_i)$ onto the image.

(3) For every $i$, $\pi_y : W^u_{loc}(c_i) \longrightarrow \mathbb{C}$ is double branched cover onto the image inside the $y-$axis. Besides, $W^u_{loc}(c_i)$ are pair-wisely disjoint.

## 2.6 Invariant holomorphic cone fields

There exists stable and unstable holomorphic cone fields, namely $C^s_x(\alpha), C^u_x(\alpha)$ inside $T^{1,0}_x\mathbb{C}^m$ for every $x \in D$, where

$$C^s_x(\alpha) := \{(\vec{v_s}, v_u) = (v_1, \cdots, v_{m-1}, v_m) \mid |v_u| \leqslant \alpha |v_s|\},$$

$$C^u_x(\alpha) := \{(\vec{v_s}, v_u) = (v_1, \cdots, v_{m-1}, v_m) \mid |v_u| \geqslant \alpha |v_s|\},$$

where $|\cdot|$ is the $L^2$ norm of a vector. Since $F_{t,a}$ is hyperbolic in $D$, there exist a $\alpha$ such that the stable and unstable cones are invariant under $F_{t,a}$.

**Proposition 2.6.1.** *If $x, f(x) \in D$, we have*

$$Df^{-1}_{f(x)}\left(C^s_{f(x)}(\alpha)\right) \subseteq C^s_x(\alpha)$$

$$Df_x\left(C^u_x(\alpha)\right) \subseteq C^u_{f(x)}(\alpha)$$

*We also have*

$$|\mu| - \epsilon \leqslant \frac{\|Df_x(v)\|}{\|v\|} \leqslant |\mu| + \epsilon, \quad \forall v \in C^u_x(\alpha) - \{0\};$$

$$\frac{\|Df^{-1}_{f(x)}(w)\|}{\|w\|} \geqslant \frac{1}{|\lambda|+\epsilon}, \quad \forall w \in C^s_{f(x)}(\alpha) - \{0\}.$$

*where* $\frac{1}{10} > \epsilon > 0$ *is a small constant satisfies* $(\lambda_{max}+\epsilon)(\mu_{max}+\epsilon)^3 < 1$, $\mu_{min} - \epsilon > 1 > \lambda_{max} + \epsilon$, $\epsilon < \frac{1}{4}(1 - \frac{1}{\mu_{min}})$, $\lambda_{min} - \epsilon > \max\limits_{2 \leq i \leq m-1} \{|\lambda_i(t,a)|\} + \epsilon$ *when* $m > 2$, $\epsilon < \frac{\mu_{min} - \lambda_{max}}{5}$ *and* $\epsilon < \frac{\lambda_{min}}{\max\limits_{t}\{|q_1(t)|+2L\}}$.

## 2.7 The foliation of Palis invariant

For a map $f$ with homoclinic or heteroclinic tangencies, Palis ([Pal78]) has introduced a differentiable invariant treating the conjugacy of diffeomorphisms. We can define a quantity of $f$ called Palis invariant , denoted as $Pa(f)$, by the following equation:

$$Pa(f) := -\frac{\log|\lambda|}{\log|\mu|}. \tag{2.7.1}$$

Since our parameter space contains $\mu, \lambda$ as coordinates, we can view Palis invariant as a map from our parameter space $P = T_\lambda \times T_\mu \times \mathbb{D}_a \times U$ to the real line. Then the level sets of Palis invariant $\{Pa(f) = c\}$ form a real codimension-1 foliation where all leaves can be written as one the forms:

$$\cup_{c \in (\lambda_{min}, \lambda_{max})} \{\mu | |\mu| = \frac{1}{c^{P(f)}}\} \times \{\lambda | |\lambda| = c\} \times \mathbb{D}_a \times U, \tag{2.7.2}$$

or

$$\cup_{c \in (\mu_{min}, \mu_{max})} \{\mu | |\mu| = c\} \times \{\lambda | |\lambda| = \frac{1}{c^{P(f)}}\} \times \mathbb{D}_a \times U. \tag{2.7.3}$$

In our case, we know that the range of $P(f)$ is just $[-\frac{\log|\lambda_{max}|}{|\mu_{max}|}, -\frac{\log|\lambda_{min}|}{|\mu_{min}|}] = [\frac{1}{\theta_0}, \frac{3}{2\theta_1}]$.

# Chapter 3

# Hyperbolic dynamics of graph transformation

**Definition 3.0.1.** For any mapping $f : A \longrightarrow B$, where $A, B$ are sets, define the **graph of** $f$, denoted by $Gr(f)$, to be the subset of $A \times B$:

$$Gr(f) := \{(x, f(x)) | x \in A\}. \tag{3.0.1}$$

We have the following proposition:

**Proposition 3.0.1.** ***A***. *Let $U$ is a connected open subset of $\mathbb{C}^k$, $f : U \longrightarrow \mathbb{C}$ a holomorphic mapping, endow $Gr(f)$ with the subspace topology, then $Gr(f)$ is a holomorphic hypersurface of $\mathbb{C}^{k+1}$ which is biholomorphic to $U$.*
***B***. *Moreover, let $f_i : U \longrightarrow \mathbb{C}$ holomorphic mappings, $i = 1, \ldots, l$ and denote $F := (f_1, \ldots, f_l) : U \longrightarrow \mathbb{C}^l$, endow $Gr(F)$ with subspace topology of $\mathbb{C}^{k+l}$, then $Gr(F)$ is a holomorphic submainfold of $\mathbb{C}^{k+l}$ which is biholomorphic to $U$.*

Denote $\mathfrak{H}(\alpha)$ be the set of all holomorphic hypersurfaces $H \subseteq D$ such that $\forall \widetilde{p} \in H, T^{1,0}_{\widetilde{p}} H \subseteq C^s_{\widetilde{p}}(\alpha)$, $\mathfrak{V}(\alpha)$ be the set of all holomorphic 1-dimensional submanifolds $V \subseteq D$ such that $\forall \widetilde{p} \in V, T^{1,0}_{\widetilde{p}} V \subseteq C^u_{\widetilde{p}}(\alpha)$. We call an element in $\mathfrak{H}(\alpha)$ **almost horizontal** while an element in $\mathfrak{V}(\alpha)$ **almost vertical**.

*Remark* 3.0.2. For every almost horizontal hypersurface $H \in \mathfrak{H}(\alpha)$, there exist a holomorphic mapping $f_H : \pi_{\vec{x}} H \longrightarrow D_y$ with bounded derivative such that $H$ can be represented as the graph of $f_H$, i.e., $H = Gr(f_H)$.

Similarly, for every almost vertical 1-dimensional submanifold $V \in \mathfrak{V}(\alpha)$, there exist a holomorphic mapping $g_V : \pi_y V \longrightarrow D_{\vec{x}}$ with bounded derivative such that $V$ can be represented as the graph of $g_V$, i.e., $V = Gr(g_V)$.

By proposition 2.6.1, we have the following lemma:

**Lemma 3.0.3.** *For any $H \in \mathfrak{H}(\alpha)$, we have $(F_{t,a}^{-1}(H) \cap D) \in \mathfrak{H}(\alpha)$; for any $V \in \mathfrak{V}(\alpha)$, we have $(F_{t,a}(V) \cap D) \in \mathfrak{V}(\alpha)$*

**Definition 3.0.2.** The two operations $(F_{t,a}^{-1}(H) \cap D) \in \mathfrak{H}(\alpha)$ and $(F_{t,a}(V) \cap D) \in \mathfrak{V}(\alpha)$ are called the graph transformation of $H$ and $D$ respectively.

Since $\{y \equiv c\} \in \mathfrak{H}(\alpha)$, $\{(x_1, \cdots, x_{m-1}) \equiv (e_1, \cdots, e_{m-1})\} \in \mathfrak{V}(\alpha)$, we get a special case:

**Lemma 3.0.4.** *If $(\vec{x_i}, y_i) = F^i(\vec{x}, y) \in D, \forall\ 0 \leqslant i \leqslant n$, then*

$$\|\vec{x_i}\| \leqslant C_0 \|\vec{x_0}\| (|\lambda| + \epsilon)^i), \quad C_1 \frac{|y_n|}{(|\mu| + \epsilon)^{n-i}} \leqslant |y_i| \leqslant C_2 \frac{|y_n|}{(|\mu| - \epsilon)^{n-i}}$$

*where $C_0, C_1, C_2$ are constants uniformly bounded away from zero.*

Estimates for orbits in $D$ have been studied previously. See for example [AŠ73]. For completeness, using the above lemma, we prove the following estimates for the norm of coordinates of points which remain in the domain of semi-linearization after $n$ iterates following [MPT20].

**Lemma 3.0.5.** *(1) If for a point $(\vec{x}, y) \in D$ with $y \neq 0$, $(\vec{x_i}, y_i) = F^i(\vec{x}, y) \in D$, $\forall\ 0 \leqslant i \leqslant n$, then for $n$ large enough we have*

$$|\vec{x}_k| \leqslant C(\vec{x}_0, y_n) |\vec{x}_0| |\lambda|^k$$

*and*

$$\frac{1}{C(\vec{x}_0, y_n)}|y_n||\mu|^{-(n-k)} \leqslant |y_k| \leqslant C(\vec{x}_0, y_n)|y_n||\mu|^{-(n-k)},$$

*where* $C(\vec{x}_0, y_n)$ *is a positive constant depending on* $\vec{x}_0, y_n$ *with a uniform upper bound* $C_1$*,i.e.,* $C(\vec{x}_0, y_n) \leqslant C_1$ *for all pairs* $(\vec{x}_0, y_n)$*, all* $n$ *and all* $F_{t,a}$*.*

*(2) When* $m \geqslant 3$*, let* $K$ *be a positive constant. Denote* $J_K = \{(x_1, x_2, \cdots, x_{m-1}, y) \in D\big||x_1| > K|(x_2, \cdots, x_{m-1})|\}$*. Then we have* $F(J_K) \cap D \subsetneq J_{\gamma K}$*, where*

$$\gamma = \frac{\lambda_{min} - \epsilon}{\lambda^- + \epsilon} > 1.$$

*Furthermore, if we have*

$$K > K_0 = \frac{1}{\sqrt{\frac{|\lambda|^2}{\epsilon^2} - 1}}, \tag{3.0.2}$$

*then for a point* $(\vec{x}, y) \in J_K$ *with* $(\vec{x_i}, y_i) = F^i(\vec{x}, y) \in J_K$*,* $\forall\ 0 \leqslant i \leqslant n$*, then for* $n$ *large enough we have*

$$|\vec{x}_k| \geqslant C(\vec{x}_0, y_n, K)|\vec{x}_0||\lambda|^k$$

*where* $C(\vec{x}_0, y_n, K)$ *is a positive constant depending on* $\vec{x}_0, y_n$ *with a uniform positive lower bound* $C_1(K)$*,i.e.,* $C(\vec{x}_0, y_n, K) \geqslant C_1(K) > 0$ *for all pairs* $(\vec{x}_0, y_n)$*, all* $n$ *and all* $F_{t,a}$*.*

*Proof.* (1) By Equation 2.5.4 and Lemma 2.5.2, since $P_s$ and $P_u$ is of order 2, we can write $P_s$ and $P_u$ by the following

$$\vec{x}_{k+1} = \Lambda\vec{x}_k + P_s(\vec{x}_k, y_k) = \Lambda\vec{x}_k + D_k(\vec{x}_k, y_k)\vec{x}_k,$$
$$y_{k+1} = \mu y_k + P_u(\vec{x}_k, y_k) = \mu y_k + E_k(\vec{x}_k, y_k)y_k,$$

where $D_k$ is a $(m-1) \times (m-1)$ matrix-valued holomorphic mappings on $(\vec{x}, y)$, $E_k$ is a holomorphic mapping on $(\vec{x}, y)$. Furthermore, by lemma 2.5.2, we have

$$||D_k|| \leq M(|\vec{x}_k| + |y_k|) \text{ and } |E_k| \leq M\,|\vec{x}_k| \tag{3.0.3}$$

for some constant $M$, where $||\cdot||$ is $L^2$ operator norm of a matrix,i.e. $||A|| := \sup\{\frac{|Av|}{|v|}\Big| v \neq 0\}$. By shrinking the domain D appropriately we may assume that

$$||D_k||, |E_k| \leq \epsilon. \tag{3.0.4}$$

By Lemma 3.0.4, we have

$$\sum_{k=0}^{n} |\vec{x}_k| \leq \sum_{k=0}^{n} (|\lambda| + \epsilon)^k \, |\vec{x}_0| \leq \frac{1}{1 - |\lambda| - \epsilon} \, |\vec{x}_0| \,, \tag{3.0.5}$$

and

$$\sum_{k=0}^{n} |y_k| \leq \sum_{k=0}^{n} \frac{1}{(|\mu| - \epsilon)^{n-k}} \, |y_n| \leq \frac{|\mu| - \epsilon}{|\mu| - \epsilon - 1} \, |y_n| \,. \tag{3.0.6}$$

Notice that the lemma is automatically true for $\vec{x}_0$ and $y_n$. Then, for some $0 \leq k \leq n$, we have

$$\vec{x}_k = \Lambda \vec{x}_{k-1} + D_{k-1}(\vec{x}_{k-1}, y_{k-1}) \vec{x}_{k-1} = \prod_{i=0}^{k-1} (\Lambda + D_i) \, \vec{x}_0, \tag{3.0.7}$$

and

$$y_k = \mu^{-1} \frac{1}{1 + \frac{E_k}{\mu}} y_{k+1} \quad = \mu^{-(n-k)} y_n \prod_{i=k}^{n-1} \left( \frac{1}{1 + \frac{E_i}{\mu}} \right). \tag{3.0.8}$$

Using (3.0.7), the fact $\ln(x) \leqslant x - 1$ for $x > 0$, (3.0.3) and (3.0.6), we have

$$\begin{aligned}
\ln |\vec{x}_k| &\leqslant \sum_{i=0}^{k-1} \ln || \, (\Lambda + D_i) \, || + \ln |\vec{x}_0| \leqslant \sum_{i=0}^{k-1} \ln(||\Lambda|| + ||D_i||) + \ln |\vec{x}_0| \\
&\leqslant \sum_{i=0}^{k-1} \ln(|\lambda| + ||D_i||) + \ln |\vec{x}_0| \leqslant \ln(|\lambda^k|) + \sum_{i=0}^{k-1} \frac{\|D_i\|}{|\lambda|} + \ln |\vec{x}_0| \\
&\leqslant \ln(|\lambda^k|) + \ln |\vec{x}_0| + \frac{M}{|\lambda|} \left( \sum_{k=0}^{n} (|\vec{x}_k| + |y_k|) \right) \\
&\leqslant \ln(|\lambda^k|) + \ln |\vec{x}_0| + \frac{M}{|\lambda|} \left( \frac{|\mu| - \epsilon}{|\mu| - \epsilon - 1} |y_n| + \frac{1}{1 - |\lambda| - \epsilon} \, |\vec{x}_0| \right).
\end{aligned}$$

Similarly, using (3.0.8), the fact $\ln(x) \leq x - 1$ for $x > 0$, (3.0.3) and (3.0.5), we have the upper bound:

$$\begin{aligned}
\ln \left| \frac{y_k}{\mu^{-(n-k)}} \right| &= \ln |y_n| + \sum_{i=k}^{n-1} \ln \frac{1}{\left| 1 + \frac{E_i}{\mu} \right|} \leqslant \ln |y_n| + \sum_{i=k}^{n-1} \frac{|E_i|/|\mu|}{1 - |E_i|/|\mu|} \\
&\leqslant \ln |y_n| + \frac{1}{|\mu| - \epsilon} \sum_{i=0}^{n} |E_i| \leqslant \ln |y_n| + \frac{M}{|\mu| - \epsilon} \left( \frac{1}{1 - |\lambda| - \epsilon} |\vec{x}_0| \right),
\end{aligned}$$

and the lower bound:

$$\ln\left|\frac{y_k}{\mu^{-(n-k)}}\right| = \ln|y_n| + \sum_{i=k}^{n-1} \ln\frac{1}{\left|1+\frac{E_i}{\mu}\right|} = \ln|y_n| - \sum_{i=k}^{n-1}\ln(\left|1+\frac{E_i}{\mu}\right|)$$

$$\geqslant \ln|y_n| - \sum_{i=k}^{n-1}\ln(1+\left|\frac{E_i}{\mu}\right|) \geqslant \ln|y_n| - \frac{1}{|\mu|}\sum_{i=0}^{n}|E_i|$$

$$\geqslant \ln|y_n| - \frac{M}{|\mu|}\left(\frac{1}{1-|\lambda|-\epsilon}|\vec{x}_0|\right).$$

Thus we finish our proof for this part.

(2) For every point $(\vec{x}, y) = (x_1, x_2, \cdots, x_{m-1}, y) \in J_K$ with $F(\vec{x}, y) \in D$, denote $(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_{m-1}, \widehat{y}) = F(\vec{x}, y)$. We also denote $\lambda^- = \max_{2\leq i\leq m-1}\{|\lambda_i(t,a)|\}$. Then by previous discussion, we have

$$\frac{|\widehat{x}_1|}{|(\widehat{x}_2, \cdots, \widehat{x}_{m-1})|} \geqslant \frac{(\lambda_{min}-\epsilon)|x_1|}{(\lambda^- + \epsilon)|(x_2, \cdots, x_{m-1})|} > \frac{\lambda_{min}-\epsilon}{\lambda^-+\epsilon}K.$$

Thus we have $F(J_K) \cap D \subsetneq J_{\gamma K}$.

Next for a point $(\vec{x}, y) \in J_K$ with $(\vec{x_i}, y_i) = F^i(\vec{x}, y) \in J_K$, $\forall\ 0 \leqslant i \leqslant n$, denote

$$(\vec{x_i}, y_i) = (x_{i,1}, x_{i,2}, \cdots, x_{i,m-1}, y_i)$$

Thus following the discussion in part (1), we have

$$x_{i+1,1} =_{i,1} + \sum_{j=1}^{m-1}(D_i)_{1j}(\vec{x_i}, y_i)x_{i,j}.$$

Thus we have

$$|x_{i+1,1}| = |_{i,1} + \sum_{j=1}^{m-1}(D_i)_{1j}(\vec{x_i}, y_i)x_{i,j}|$$

$$\geqslant |_{i,1}| - |\sum_{j=1}^{m-1}(D_i)_{1j}(\vec{x_i}, y_i)x_{i,j}|$$

$$\geqslant |_{i,1}| - \sqrt{\sum_{j=1}^{m-1}|(D_i)_{1j}(\vec{x_i}, y_i)|^2} \cdot \sqrt{\sum_{j=1}^{m-1}|x_{i,j}|^2}$$

$$> |_{i,1}| - \sqrt{\sum_{j=1}^{m-1}|(D_i)_{1j}(\vec{x_i}, y_i)|^2} \cdot \sqrt{1+\frac{1}{K^2}}|x_{i,1}|$$

$$= (|\lambda| - \sqrt{1+\frac{1}{K^2}}\sqrt{\sum_{j=1}^{m-1}|(D_i)_{1j}(\vec{x_i}, y_i)|^2})|x_{i,1}|.$$

By condition (3.0.2), we have

$$|\lambda|-\sqrt{1+\frac{1}{K^2}}\sqrt{\sum_{j=1}^{m-1}|(D_i)_{1j}(\vec{x_i},y_i)|^2}>|\lambda|-\sqrt{1+\frac{1}{K^2}}\epsilon>0.$$

For any $1\leqslant k\leqslant n$, using inequality $\ln(1-x)\geqslant\frac{-x}{1-x}$ where $x\in(0,1)$, we have the following estimate:

$$\begin{aligned}
\ln(|x_{k,1}|) &= \ln(|\lambda|^k)+\ln(|x_{0,1}|)+\sum_{i=0}^{k-1}\ln(1-\frac{\sqrt{1+\frac{1}{K^2}}}{|\lambda|}\sqrt{\sum_{j=1}^{m-1}|(D_i)_{1j}(\vec{x_i},y_i)|^2})\\
&\geqslant \ln(|\lambda|^k)+\ln(|x_{0,1}|)-\sum_{i=0}^{k-1}\frac{\frac{\sqrt{1+\frac{1}{K^2}}}{|\lambda|}\sqrt{\sum_{j=1}^{m-1}|(D_i)_{1j}(\vec{x_i},y_i)|^2}}{1-\frac{\sqrt{1+\frac{1}{K^2}}}{|\lambda|}\sqrt{\sum_{j=1}^{m-1}|(D_i)_{1j}(\vec{x_i},y_i)|^2}}\\
&> \ln(|\lambda|^k)+\ln(|x_{0,1}|)-\frac{\sqrt{1+\frac{1}{K^2}}}{|\lambda|-\epsilon\sqrt{1+\frac{1}{K^2}}}\sum_{i=0}^{k-1}\sqrt{\sum_{j=1}^{m-1}|(D_i)_{1j}(\vec{x_i},y_i)|^2}\\
&> \ln(|\lambda|^k)+\ln(|x_{0,1}|)-\frac{\sqrt{1+\frac{1}{K^2}}}{|\lambda|-\epsilon\sqrt{1+\frac{1}{K^2}}}\sum_{i=0}^{k-1}M(|\vec{x_i}|+|y_i|)\\
&\geqslant \ln(|\lambda|^k)+\ln(|x_{0,1}|)-\frac{\sqrt{1+\frac{1}{K^2}}}{|\lambda|-\epsilon\sqrt{1+\frac{1}{K^2}}}M\left(\frac{|\mu|-\epsilon}{|\mu|-\epsilon-1}|y_n|+\frac{1}{1-|\lambda|-\epsilon}|\vec{x_0}|\right).
\end{aligned}$$

Finally we have

$$\begin{aligned}
\ln(|\vec{x_k}|)-\ln(|\vec{x_0}|)-\ln(|\lambda|^k) &> \ln(|x_{k,1}|)-\ln(\sqrt{1+\frac{1}{K^2}}|x_{0,1}|)-\ln(|\lambda|^k)\\
&> -\ln(\sqrt{1+\frac{1}{K^2}})-\frac{\sqrt{1+\frac{1}{K^2}}}{|\lambda|-\epsilon\sqrt{1+\frac{1}{K^2}}}\sum_{i=0}^{k-1}M(|\vec{x_i}|+|y_i|).
\end{aligned}$$

□

*Remark* 3.0.6. By part (2) of the lemma, for $q_1(t)$, it will enter $J_{K_0}$ in finite forward iterations. Thus by replacing $q_1(t)$ with some $F^s_{t,0}(q_t)$ where $s>0$, we can assume $q_1(t)\in J_{K_0}$.

For a point $Z=(\vec{Z}_x,Z_y)\in D$, denote the linear unstable cone with slope $\iota>0$ by

$$V^u(Z;\iota):=\{(\vec{x},y)\in D\big||y-Z_y|\geqslant\iota||\vec{x}-\vec{Z}_x||\}.$$

Now consider 2 distinct points $(\vec{x}_0, y_0), (\vec{x}'_0, y'_0) \in D$ with $F^k((\vec{x}_0, y_0)) \in D$ and $F^k((\vec{x}'_0, y'_0)) \in D$ for $0 \leqslant k \leqslant n$. Denote their forward iterations by $(\vec{x}_k, y_k) = F^k((\vec{x}_0, y_0))$, $(\vec{x}'_k, y'_k) = F^k((\vec{x}'_0, y'_0))$. Then we have the following estimation:

**Lemma 3.0.7.** *Suppose there exists an $\iota > 0$ such that $(\vec{x}'_0, y'_0) \in V^u((\vec{x}_0, y_0); \iota)$. Then there exist a constant $\zeta = \max\{0, -\log_{|\mu|} \frac{\iota(|\mu|-|\lambda|-3\epsilon)}{C_1 ML}\} \geqslant 0$ not depending on $n$, such that when $n$ large enough, for $0 \leqslant k \leqslant n - \zeta$, $(\vec{x}'_k, y'_k) \in V^u((\vec{x}_k, y_k); \iota^-)$, where $\iota^- = \min\{\iota, \frac{|\mu|-|\lambda|-3\epsilon}{2\epsilon}\} > 0$.*

*Furthermore, there exists an uniform constant $C((\vec{x}_0, y_n, \vec{x}'_0, y'_n, \iota)) > 0$ such that*

$$|y'_k - y_k| > C((\vec{x}_0, y_n, \vec{x}'_0, y'_n, \iota))|\mu|^k |y'_0 - y_0|. \tag{3.0.9}$$

*for any $0 \leqslant k \leqslant n$. $C((\vec{x}_0, y_n, \vec{x}'_0, y'_n, \iota))$ has a uniform positive lower bound $C(\iota)$ only depending on $\iota$.*

*Proof.* Again by previous lemma, we have the following:

$$\begin{aligned}
\vec{x}_{k+1} &= \Lambda \vec{x}_k + P_s(\vec{x}_k, y_k) = \Lambda \vec{x}_k + D_k(\vec{x}_k, y_k)\vec{x}_k, \\
y_{k+1} &= \mu y_k + P_u(\vec{x}_k, y_k) = \mu y_k + E_k(\vec{x}_k, y_k) y_k, \\
\vec{x}'_{k+1} &= \Lambda \vec{x}'_k + P_s(\vec{x}'_k, y'_k) = \Lambda \vec{x}'_k + D_k(\vec{x}'_k, y'_k)\vec{x}'_k, \\
y'_{k+1} &= \mu y'_k + P_u(\vec{x}'_k, y'_k) = \mu y'_k + E_k(\vec{x}'_k, y'_k) y'_k.
\end{aligned}$$

By lemma 2.5.2, we have

$$||D_k(\vec{x}_k, y_k) - D_k(\vec{x}'_k, y'_k)|| \leq M(|\vec{x}_k - \vec{x}'_k| + |y_k - y'_k|) \tag{3.0.10}$$

and

$$|E_k(\vec{x}_k, y_k) - E_k(\vec{x}'_k, y'_k)| \leq M \left|\vec{x}_k - \vec{x}'_k\right|. \tag{3.0.11}$$

Then we have

$$\begin{aligned}
|y'_{k+1} - y_{k+1}| &= |\mu(y'_k - y_k) + E_k(\vec{x}'_k, y'_k)y'_k - E_k(\vec{x}_k, y_k)y_k| \\
&\geqslant |\mu||y'_k - y_k| - |E_k(\vec{x}'_k, y'_k)y'_k - E_k(\vec{x}_k, y_k)y_k| \\
&\geqslant |\mu||y'_k - y_k| - |E_k(\vec{x}'_k, y'_k)y'_k - E_k(\vec{x}'_k, y'_k)y_k| - |E_k(\vec{x}'_k, y'_k)y_k - E_k(\vec{x}_k, y_k)y_k| \\
&\geqslant (|\mu| - |E_k(\vec{x}'_k, y'_k)|)|y'_k - y_k| - M\left|\vec{x}_k - \vec{x}'_k\right||y_k|.
\end{aligned}$$

and

$$\begin{aligned}
\left|\vec{x}'_{k+1}-\vec{x}_{k+1}\right| &= |\Lambda(\vec{x}'_k-\vec{x}_k)+D_k(\vec{x}'_k,y'_k)\vec{x}'_k-D_k(\vec{x}_k,y_k)\vec{x}_k| \\
&\leqslant (|\lambda|+|D_k(\vec{x}'_k,y'_k)|)|\vec{x}'_k-\vec{x}_k|+||D_k(\vec{x}_k,y_k)-D_k(\vec{x}'_k,y'_k)||\cdot|\vec{x}_k| \\
&\leqslant (|\lambda|+|D_k(\vec{x}'_k,y'_k)|)|\vec{x}'_k-\vec{x}_k|+M(|\vec{x}_k-\vec{x}'_k|+|y_k-y'_k|)|\vec{x}_k|.
\end{aligned}$$

If we denote $l_k>0$ be a number such that $(\vec{x}'_k,y'_k)\in V^u((\vec{x}_k,y_k);l_k)$, we let $l_0=\iota$. Then we have

$$\begin{aligned}
|y'_{k+1}-y_{k+1}| &\geqslant (|\mu|-|E_k(\vec{x}'_k,y'_k)|)|y'_k-y_k|-M\,|\vec{x}_k-\vec{x}'_k|\,|y_k| \\
&\geqslant (|\mu|-|E_k(\vec{x}'_k,y'_k)|-\frac{M|y_k|}{l_k})|y'_k-y_k|,
\end{aligned}$$

and

$$\begin{aligned}
\left|\vec{x}'_{k+1}-\vec{x}_{k+1}\right| &\leqslant (|\lambda|+|D_k(\vec{x}'_k,y'_k)|)|\vec{x}'_k-\vec{x}_k|+M(|\vec{x}_k-\vec{x}'_k|+|y_k-y'_k|)|\vec{x}_k| \\
&\leqslant (\frac{|\lambda|+|D_k(\vec{x}'_k,y'_k)|+M|\vec{x}_k|}{l_k}+M|\vec{x}_k|)|y'_k-y_k|.
\end{aligned}$$

Overall we have the following inequality

$$\begin{aligned}
l_{k+1} &\geqslant \frac{(|\mu|-|E_k(\vec{x}'_k,y'_k)|)l_k-M|y_k|}{|\lambda|+|D_k(\vec{x}'_k,y'_k)|+(1+l_k)M|\vec{x}_k|} \\
&\geqslant \frac{(|\mu|-\epsilon)l_k-M|y_k|}{|\lambda|+\epsilon+(1+l_k)M|\vec{x}_k|}.
\end{aligned}$$

Denote $R_{k,n}(l)=\frac{(|\mu|-\epsilon)l-M|y_k|}{|\lambda|+\epsilon+(1+l)M|\vec{x}_k|}$ Then function $t=R_{k,n}(l)$ has following properties:

(1) It is increasing and has two asymptotes: $l=-\frac{|\lambda|+\epsilon}{M|\vec{x}_k|}-1$ and $t=\frac{M|y_k|}{|\mu|-\epsilon}$.

(2) It has two intersection points with the line $t=l$, denoted by $(w^-_{k,n},w^-_{k,n})$ and $(w^+_{k,n},w^+_{k,n})$, where

$$w^{\pm}_{k,n}=\frac{(|\mu|-|\lambda|-2\epsilon-M|\vec{x}_k|)\pm\sqrt{(|\mu|-|\lambda|-2\epsilon-M|\vec{x}_k|)^2-4M^2|\vec{x}_k|\cdot|y_k|}}{2M|\vec{x}_k|}$$

are the two roots of the quadratic equation:

$$M|\vec{x}_k|l^2-(|\mu|-|\lambda|-2\epsilon-M|\vec{x}_k|)l+M|y_k|=0.$$

Since we have $M|\vec{x}_k|, M|y_k| < \epsilon < \frac{|\mu|-|\lambda|}{5}$, thus the discriminant of the quadratic equation

$$(|\mu| - |\lambda| - 2\epsilon - M|\vec{x}_k|)^2 - 4M^2|\vec{x}_k| \cdot |y_k| > (|\mu| - |\lambda| - 3\epsilon)^2 - 4\epsilon^2 > 0.$$

is always positive. We have the inequality:

$$0 < w^-_{k,n} < \frac{|\mu| - |\lambda| - 2\epsilon - M|\vec{x}_k|}{2M|\vec{x}_k|} < w^+_{k,n}.$$

(3). When $l \in (w^-_{k,n}, w^+_{k,n})$, we have $R_{k,n}(l) > l$. When $l \in (w^+_{k,n}, +\infty)$, we have $R_{k,n}(l) > w^+_{k,n}$. Overall, when $l \in (w^-_{k,n}, +\infty)$, we have $R_{k,n}(l) \geqslant \min\{l, w^+_{k,n}\}$.

Now when $n$ large enough, by lemma 3.0.5, we have

$$\begin{aligned}
w^-_{k,n} &= \frac{(|\mu| - |\lambda| - 2\epsilon - M|\vec{x}_k|) - \sqrt{(|\mu| - |\lambda| - 2\epsilon - M|\vec{x}_k|)^2 - 4M^2|\vec{x}_k| \cdot |y_k|}}{2M|\vec{x}_k|} \\
&= \frac{2M|y_k|}{(|\mu| - |\lambda| - 2\epsilon - M|\vec{x}_k|) + \sqrt{(|\mu| - |\lambda| - 2\epsilon - M|\vec{x}_k|)^2 - 4M^2|\vec{x}_k| \cdot |y_k|}} \\
&< \frac{M|y_k|}{|\mu| - |\lambda| - 2\epsilon - M|\vec{x}_k|} \\
&\leqslant \frac{M|y_k|}{|\mu| - |\lambda| - 3\epsilon} \\
&\leqslant \frac{C_1 M|y_n||\mu|^{-(n-k)}}{|\mu| - |\lambda| - 3\epsilon} \\
&\leqslant \frac{C_1 ML|\mu|^{-(n-k)}}{|\mu| - |\lambda| - 3\epsilon}
\end{aligned}$$

where $L$ is the radius of $D$. Thus when $k \leqslant n + \min\{0, \log_{|\mu|} \frac{\iota(|\mu|-|\lambda|-3\epsilon)}{C_1 ML}\}$, we have

$$w^-_{k,n} < \frac{C_1 ML|\mu|^{-(n-k)}}{|\mu| - |\lambda| - 3\epsilon} < \iota.$$

Besides, for any $0 \leqslant k, k' \leqslant n$, we have

$$\begin{aligned}
w^-_{k,n} &< \frac{M|y_k|}{|\mu| - |\lambda| - 2\epsilon - M|\vec{x}_k|} \\
&\leqslant \frac{\epsilon}{|\mu| - |\lambda| - 3\epsilon} \\
&< \frac{|\mu| - |\lambda| - 3\epsilon}{2\epsilon} \\
&< \frac{|\mu| - |\lambda| - 2\epsilon - M|\vec{x}_{k'}|}{2M|\vec{x}_{k'}|} \\
&< w^+_{k',n}.
\end{aligned}$$

Thus we let $\zeta = \max\{0, -\log_{|\mu|} \frac{\iota(|\mu|-|\lambda|-3\epsilon)}{C_1 ML}\} \geqslant 0$, then when $n$ large enough, for any $1 \leqslant k \leqslant n-\zeta$, we have the following estimations:

$$
\begin{aligned}
l_k &\geqslant R_{k-1,n}(l_{k-1}) \\
&\cdots \\
&\geqslant R_{k-1,n} \circ \cdots \circ R_{0,n}(l_0) \\
&\geqslant R_{k-1,n} \circ \cdots \circ R_{1,n}(\min\{\iota, w_{0,n}^+\}) \\
&\cdots \\
&\geqslant \min\{\iota, w_{0,n}^+, \cdots, w_{k-1,n}^+\} \\
&\geqslant \min\{\iota, \frac{|\mu|-|\lambda|-3\epsilon}{2\epsilon}\} \\
&= \iota^-.
\end{aligned}
$$

Thus we know $(\vec{x}'_k, y'_k) \in V^u((\vec{x}_k, y_k); \iota^-)$.

Finally, for any $1 \leqslant k \leqslant n-\zeta$, we have

$$
\begin{aligned}
|y'_k - y_k| &\geqslant (|\mu| - |E_{k-1}(\vec{x}'_{k-1}, y'_{k-1})| - \frac{M|y_{k-1}|}{l_{k-1}}|)|y'_{k-1} - y_{k-1}| \\
&\geqslant (|\mu| - |E_{k-1}(\vec{x}'_{k-1}, y'_{k-1})| - \frac{M|y_{k-1}|}{l_{k-1}})|y'_{k-1} - y_{k-1}| \\
&\geqslant (|\mu| - |E_{k-1}(\vec{x}'_{k-1}, y'_{k-1})| - \frac{M|y_{k-1}|}{\iota^-})|y'_{k-1} - y_{k-1}| \\
&\geqslant |y'_0 - y_0| \prod_{i=0}^{k-1} (|\mu| - |E_i(\vec{x}'_i, y'_i)| - \frac{M|y_i|}{\iota^-}).
\end{aligned}
$$

Then we have

$$\begin{aligned}
\ln|y'_k - y_k| &\geqslant \ln|y'_0 - y_0| + \sum_{i=0}^{k-1} \ln\left(|\mu| - |E_i(\vec{x}'_i, y'_i)| - \frac{M|y_i|}{\iota^-}\right) \\
&\geqslant \ln|y'_0 - y_0| + k\ln|\mu| + \sum_{i=0}^{k-1} \ln\left(1 - \frac{|E_i(\vec{x}'_i, y'_i)|}{|\mu|} - \frac{M|y_i|}{|\mu|\iota^-}\right) \\
&\geqslant \ln|y'_0 - y_0| + k\ln|\mu| - \sum_{i=0}^{k-1} \frac{\frac{|E_i(\vec{x}'_i, y'_i)|}{|\mu|} + \frac{M|y_i|}{|\mu|\iota^-}}{1 - \frac{|E_i(\vec{x}'_i, y'_i)|}{|\mu|} - \frac{M|y_i|}{|\mu|\iota^-}} \\
&\geqslant \ln|y'_0 - y_0| + k\ln|\mu| - \frac{|\mu|}{|\mu| - \epsilon(1 + \frac{1}{\iota^-})} \sum_{i=0}^{k-1} \left(\frac{|E_i(\vec{x}'_i, y'_i)|}{|\mu|} + \frac{M|y_i|}{|\mu|\iota^-}\right) \\
&\geqslant \ln|y'_0 - y_0| + k\ln|\mu| - \frac{|\mu|}{|\mu| - \epsilon(1 + \frac{1}{\iota^-})} \sum_{i=0}^{k-1} \left(\frac{M|\vec{x}'_i|}{|\mu|} + \frac{M|y_i|}{|\mu|\iota^-}\right) \\
&\geqslant \ln|y'_0 - y_0| + k\ln|\mu| - \frac{|\mu|}{|\mu| - \epsilon(1 + \frac{1}{\iota^-})} \left(\frac{M}{|\mu|} \frac{1}{1 - |\lambda| - \epsilon} |\vec{x}'_0| + \frac{M}{|\mu|\iota^-} \frac{|\mu| - \epsilon}{|\mu| - \epsilon - 1} |y_n|\right).
\end{aligned}$$

Thus

$$|y'_k - y_k| > C((\vec{x}_0, y_n, \vec{x}'_0, y'_n, \iota))|\mu|^k |y'_0 - y_0|$$

for any $0 \leqslant k \leqslant n - \zeta. C((\vec{x}_0, y_n, \vec{x}'_0, y'_n, \iota))$ have a lower bound:

$$\begin{aligned}
C((\vec{x}_0, y_n, \vec{x}'_0, y'_n, \iota)) &= \exp\left[-\frac{|\mu|}{|\mu| - \epsilon(1 + \frac{1}{\iota^-})} \left(\frac{M}{|\mu|} \frac{1}{1 - |\lambda| - \epsilon} |\vec{x}'_0| + \frac{M}{|\mu|\iota^-} \frac{|\mu| - \epsilon}{|\mu| - \epsilon - 1} |y_n|\right)\right] \\
&\geqslant \exp\left[-\frac{|\mu|}{|\mu| - \epsilon(1 + \frac{1}{\iota^-})} \left(\frac{M}{|\mu|} \frac{1}{1 - |\lambda| - \epsilon} L + \frac{M}{|\mu|\iota^-} \frac{|\mu| - \epsilon}{|\mu| - \epsilon - 1} L\right)\right] > 0.
\end{aligned}$$

When $n$ large enough, $\zeta \ll n$, we can see that for $k \geqslant n - \zeta$,

$$\begin{aligned}
|y'_{n-\zeta+1} - y_{n-\zeta+1}| &\geqslant (|\mu| - |E_{n-\zeta}(\vec{x}'_{n-\zeta}, y'_{n-\zeta})|)|y'_{n-\zeta} - y_{n-\zeta}| - M\left|\vec{x}_{n-\zeta} - \vec{x}'_{n-\zeta}\right| |y_{n-\zeta}| \\
&> |y'_{n-\zeta} - y_{n-\zeta}|
\end{aligned}$$

since

$$|y'_{n-\zeta} - y_{n-\zeta}| > C((\vec{x}_0, y_n, \vec{x}'_0, y'_n, \iota))|\mu|^{n-\zeta}|y'_0 - y_0|$$

and

$$\left|\vec{x}_{n-\zeta} - \vec{x}'_{n-\zeta}\right| |y_{n-\zeta}| < C|\lambda|^{n-\zeta}|\mu|^{-\zeta} \ll |\mu|^{n-\zeta}.$$

Repeat this process for $k \leqslant n$, by shrinking the constant $C((\vec{x}_0, y_n, \vec{x}'_0, y'_n, \iota))$ properly, we have

$$|y'_k - y_k| > C((\vec{x}_0, y_n, \vec{x}'_0, y'_n, \iota))|\mu|^k |y'_0 - y_0|.$$

for any $0 \leqslant k \leqslant n$. $\square$

When orbit of points are inside the hyperbolic domain $D$, we may extend lemma 17 in [BMP18] in our situation to get a estimation on the differential matrix:

**Lemma 3.0.8.** *If $(x, y) \in \mathbb{D}^{m-1} \times \mathbb{D}$ and $F^i(x, y) \in \mathbb{D}^{m-1} \times \mathbb{D}$, for all $i \leq n$, then for any $k \leqslant n$,we have*

$$DF^k(x, y) = \begin{pmatrix} a_{11}(k)\lambda^k \mu^k & a_{12}(k)\lambda^k \mu^k \\ a_{21}(k)\frac{1}{\mu^{n-k}} & a_{22}(k)\mu^k \end{pmatrix} \tag{3.0.12}$$

*where $a_{ij}(k)$ are matrices satisfy following properties:*

*(1) $a_{ij}(k)$ are matrices holomorphically depending on all the parameters and unifomrly bounded with respect to $n, k$;*

*(2) there exists constant $\alpha \in (\frac{1}{|\mu_{min}|}, 1)$ and $C > 0$ such that for any integers $i, j$ from $1$ to $m-1$, $|a_{11}(k)_{ij}| < C\alpha^k$;*

*(3) $a_{22}(k) \neq 0$ and uniformly away from zero.*

*Proof.* Let us denote $(x_k, y_k) := F^k(x, y)$ and by equation (2.5.4) and lemma 2.5.2, we can denote the differential matrix at this point as

$$DF(x_k, y_k) = \begin{pmatrix} \Lambda + D_{11}(k) & D_{12}(k) \\ D_{21}(k) & \mu + D_{22}(k) \end{pmatrix} \tag{3.0.13}$$

and we have the following estimations on each entity of every matrices:

$$||D_{11}(k)||_{max} = O(|x_k| + |y_k|), \tag{3.0.14}$$

$$||D_{12}(k)||_{max} = O(|x_k|), \tag{3.0.15}$$

$$||D_{21}(k)||_{max} = O(|y_k|), \tag{3.0.16}$$

$$|D_{22}(k)| = O(|x_k|). \tag{3.0.17}$$

where $||\cdot||_{max}$ denote the max norm of a matrix. Now let

$$DF^k(x,y) = \begin{pmatrix} a_{11}(k)\lambda^k\mu^k & a_{12}(k)\lambda^k\mu^k \\ a_{21}(k)\frac{1}{\mu^{n-k}} & a_{22}(k)\mu^k \end{pmatrix},$$

then by (3.0.13), we have

$$a_{11}(k+1) = \frac{1}{\lambda\mu}\left(\Lambda + D_{11}(k)\right)a_{11}(k) + D_{12}(k)a_{21}(k)\frac{1}{\lambda^{k+1}}\frac{1}{\mu^{n+1}}, \tag{3.0.18}$$

and

$$a_{21}(k+1) = \lambda^k\mu^{n-1}D_{21}(k)a_{11}(k) + \left(1 + \frac{1}{\mu}D_{22}(k)\right)a_{21}(k) \tag{3.0.19}$$

Since $x_k = O(|\lambda^k|)$, we have

$$|D_{22}(k)| < C|\lambda^k| \tag{3.0.20}$$

for some constant $C > 0$ dependent of $k, n$, thus

$$\left|\prod_{i<k}\left(1 + \frac{1}{\mu}D_{22}(i)\right)\right| \leqslant \prod_{i=0}^{n}\left(1 + \frac{C}{|\mu|}|\lambda|^i\right) < \prod_{i=0}^{\infty}\left(1 + \frac{C}{|\mu|}|\lambda|^i\right) < C' \tag{3.0.21}$$

where $C'$ independent of $n, k$. Then from (3.0.19), (3.0.21) and the fact $a_{21}(0) = 0$, we have

$$||a_{21}(k)||_{max} = O\left(\sum_{i<k}(\lambda\mu)^i\,||a_{11}(i)||_{max}\right) \tag{3.0.22}$$

Let $M_k = \max_{i\leqslant k}||a_{11}(i)||_{max}$. Then by (3.0.18) and (3.0.22), we have

$$\begin{aligned} ||a_{11}(k+1)||_{max} &\leqslant (\frac{1}{|\mu|} + O(|x_k| + |y_k|))||a_{11}(k)||_{max} + \frac{1}{|\mu|^n}O\left(\sum_{i<k}(\lambda\mu)^i\,||a_{11}(i)||_{max}\right) \\ &\leqslant [\frac{1}{|\mu|} + C_1(|\lambda|^k + \frac{1}{|\mu|^{n-k}})]||a_{11}(k)||_{max} + \frac{1}{|\mu|^n}\frac{C_2}{1-|\lambda\mu|}M_{k-1} \\ &\leqslant [\frac{1}{|\mu|} + C_1(|\lambda|^k + \frac{1}{|\mu|^{n-k}}) + \frac{1}{|\mu|^n}\frac{C_2}{1-|\lambda\mu|}]M_k, \end{aligned}$$

where $C_{1,2} > 0$ are constants independent of $n$ and $k$. Thus when $n$ large enough, fix $\alpha_{1,2} \in (\frac{1}{|\mu_{min}|}, 1)$ with $\alpha_1 < \alpha_2$, there exist positive integers $k_0$ and $k_1$ independent of $n$, such that, for any $k$ with $k_0 \leqslant k \leqslant n - k_1$, we have

$$||a_{11}(k+1)||_{max} \leqslant \alpha_1||a_{11}(k)||_{max} + \frac{1}{|\mu|^n}\frac{C_2}{1-|\lambda\mu|}M_{k-1} \leqslant \alpha_2 M_k, \tag{3.0.23}$$

and for $0 \leqslant k < k_0$ or $n - k_1 < k \leqslant n$, we have

$$||a_{11}(k+1)||_{max} \leqslant (\alpha_2 + 2C_1)M_k. \tag{3.0.24}$$

Thus we know when $n$ large enough, $M_k$ are uniformly bounded for all $k$.

Furthermore, we have when $k_0 \leqslant k \leqslant n - k_1$, there exist some constant $C_3 > 0$ such that

$$||a_{11}(k+1)||_{max} \leqslant \alpha_1 ||a_{11}(k)||_{max} + C_3 \frac{1}{|\mu|^n}. \tag{3.0.25}$$

Then we know that

$$||a_{11}(k)||_{max} \leqslant (\alpha_1)^{k-k_0} ||a_{11}(k_0)||_{max} + \frac{1}{|\mu|^n} \frac{C_3}{1-\alpha_1}. \tag{3.0.26}$$

Thus by the choice of $\alpha_1$, we know there exists constant $C > 0$ such that when $n$ large enough, for every $0 \leqslant k \leqslant n$, we have

$$||a_{11}(k)||_{max} \leqslant C(\alpha_1)^k. \tag{3.0.27}$$

By (3.0.22), we also get $||a_{21}(k)||_{max}$ are uniformly bounded for all $k$.

Similarly, we have

$$a_{12}(k+1) = \frac{1}{\lambda\mu} \left(\Lambda + D_{11}(k)\right) a_{12}(k) + \frac{1}{\lambda^k} D_{12}(k) \frac{1}{\lambda\mu} a_{22}(k), \tag{3.0.28}$$

and

$$a_{22}(k+1) = \frac{\lambda^k}{\mu} D_{21}(k) a_{12}(k) + \left(1 + \frac{D_{22}(k)}{\mu}\right) a_{22}(k). \tag{3.0.29}$$

By (3.0.21), (3.0.29) and the fact $a_{22}(0) = 1$, we get

$$|a_{22}(k)| = O\left(1 + \frac{1}{|\mu|^n} \sum_{i<k} (\lambda\mu)^i \, ||a_{12}(i)||_{max}\right). \tag{3.0.30}$$

Let $M'_k = \max_{i \leqslant k} ||a_{12}(i)||_{max}$. Then by (3.0.28), we have

$$\begin{aligned} ||a_{12}(k+1)||_{max} &\leqslant (\frac{1}{|\mu|} + O(|x_k| + |y_k|))||a_{12}(k)||_{max} + O\left(1 + \frac{1}{|\mu|^n} \sum_{i<k} (\lambda\mu)^i \, ||a_{12}(i)||_{max}\right) \\ &\leqslant [\frac{1}{|\mu|} + C_3(|\lambda|^k + \frac{1}{|\mu|^{n-k}}) + \frac{1}{|\mu|^n} \frac{C_4}{1 - |\lambda\mu|}] M'_k + C_5, \end{aligned}$$

where $C_{3,4,5} > 0$ are constants independent of $n$ and $k$. By same argument as above, when $n$ large enough, we can settle down that $|a_{12}(k)||_{max}$ are bounded for all $k \leqslant n$ and this bound are independent of $k, n$. Thus by (3.0.30), when $n$ large enough, we know $a_{22}(k)$ are uniformly bounded and away from zero for all $k$. Thus we finished the proof. $\square$

# Chapter 4

# Cascades of sinks

By shrinking the coordinates and replacing $N$ with $N+n$, we may assume

$$\pi_y F_{t,0}^N(W_{\mathrm{loc}}^u(q_3(t,0))) \supseteq D_y(0,2)$$

and

$$dist(W_{\mathrm{loc}}^u(q_3(t,0)), \mathbb{C} - D_y(0,2)) > 10\epsilon.$$

We will prove following proposition in this section:

**Proposition 4.0.1.** *For $n$ large enough, there exists a holomorphic mapping $a_n : T_\mu(\mu_{min}, \mu_{max}) \times T_\lambda(\lambda_{min}, \lambda_{max}) \times U \longrightarrow \mathbb{D}_a(2)$ and a positive constant $\eta > 0$ such that there exists a disc in in the a-parameter with center $a_n(t)$ and radius $\frac{\eta}{|\mu_{max}|^{2n}}$, which we denoted by $D_a(a_n(t), \frac{\eta}{|\mu_{max}|^{2n}}) \subset \mathbb{D}_a(2)$, such that they satisfy the following:*
*For each $a \in D_a(a_n(t), \frac{\eta}{|\mu_{max}|^{2n}})$, $F_{t,a}$ has a sink with period $N+n$. Moreover, the sink created is holomorphically depending on $t$.*

Just a reminder, as we mentioned previously, $t$ is just $(\mu, \lambda, \tau)$.

We will prove this theorem by construction in several steps, throughout the process,$t = (\mu, \lambda, \tau)$ will be fixed:

**Step 1.** Since by definition, we know

$$(D_y, W^u_{loc,q_3(t)}(p), \pi_y F^N_{t,0})$$

is a quadratic-like map, then there exists a simply connected domain $D_0(t,0) \subseteq W^u_{loc,q_3(t)}(p)$ such that

$$\pi_y F^N_{t,0}(D_0(t,0)) = D_y(0,2),$$

$\mathrm{dist}(\partial D_0(t,0), q_3(t)) = m$, where $m$ is a constant satisfying

$$0 < m < \frac{1}{2}. \tag{4.0.1}$$

Besides, $(D_y(0,2), D_0(t,0), \pi_y F^N_{t,0})$ is a quadratic-like map, with unique critical point $q_3(t)$ and critical value 0.

*Remark* 4.0.2. The three lemmas 4.0.3,4.0.4,4.0.6 extends above fact further, and each lemma can be viewed as an extension from the previous lemma. Even though we state these lemmas in a similar way, the proofs for each lemma use different ingredients.

Now we first prove the following lemma to show similar construction exists for every $n$:

**Lemma 4.0.3.** *There exist a sequences of simply-connected domains $D_n(t,0)$ with $q_3(t) = \bigcap_n D_n(t,0)$ and $D_n(t,0) \subseteq D_{n-1}(t,0)$ such that $(D_y(0,2), D_n(t,0), \pi_y F^{N+n}_{t,0})$ is a quadratic-like map, with unique critical point $q_3(t)$ and critical value 0. For n large enough, the diameter of $D_n(t,0)$ has the following estimates:*

$$\frac{1}{C_2|\mu|^{\frac{n}{2}}} \leqslant \mathrm{diam}(D_n(t,0)) \leqslant \frac{C_2}{|\mu|^{\frac{n}{2}}},$$

*where $C_2$ is a positive constant.*

*Proof.* We construct $D_n(t,0)$ inductively on n. Suppose the Lemma is verified from 0 to $n$, then by Lemma 3.0.5, we know for every point $(0,z) \in D_n(t,0)$,

$$\begin{aligned}
\left|\frac{\pi_y F_{t,0}^{N+n+1}(0,z)}{\mu} - \pi_y F_{t,0}^{N+n}(0,z)\right| &= \left|\left(\frac{\pi_y F_{t,0}}{\mu} - \pi_y\right)(F_{t,0}^{N+n}(0,z))\right| \\
&= |E_n(F_{t,0}^{N+n}(0,z))\pi_y(F_{t,0}^{N+n}(0,z))| \\
&\leqslant \epsilon|\pi_y(F_{t,0}^{N+n}(0,z))| \\
&\leqslant 2\epsilon \\
&< \mathrm{d}(D_0(t,0), \mathbb{C} - D_y(0,2)) \\
&< \mathrm{d}(D_n(t,0), \mathbb{C} - D_y(0,2)).
\end{aligned}$$

Thus, by Proposition 2.2.4, we know $(\frac{\pi_y F_{t,0}^{N+n+1}}{\mu})^{-1}(D_y(0, 2-2\epsilon))$ is a simply connected domain inside $D_n(t,0)$.

Thus we know

$$(D_y(0, 2-2\epsilon), (\frac{\pi_y F_{t,0}^{N+n+1}}{\mu})^{-1}(D_y(0,2-2\epsilon)), \frac{\pi_y F_{t,0}^{N+n+1}}{\mu})$$

is a quadratic-like map. Since we know 0 has only 1 preimage $q_3(t)$, we know the unique critical point is $q_3(t)$ and corresponding critical point 0.

Thus we can see that

$$(D_y(0, |\mu|(2-2\epsilon)), (\pi_y F_{t,0}^{N+n+1})^{-1}(D_y(0, |\mu|(2-2\epsilon))), \pi_y F_{t,0}^{N+n+1})$$

is a quadratic-like map.

Since $|\mu|(2-2\epsilon) > 2$ and $0 \in D_y(0,2)$. We conclude that

$$(D_y(0,2), (\pi_y F_{t,0}^{N+n+1})^{-1}(D_y(0,2)), \pi_y F_{t,0}^{N+n+1})$$

is a quadratic-like map.

Then we define $D_{n+1}(t,0)$ to be $(\pi_y F_{t,0}^{N+n+1})^{-1}(D_y(0,2))$, a simply connected domain inside $D_n(t,0)$. Thus we finish the induction step and the sequence $D_n(t,0)$ is constructed.

Now we estimate the diameter of $D_n(t,0)$. Since

$$F_{t,0}^{N}(D_n(t,0)) = F_{t,0}^{-n}(F_{t,0}^{N+n}(D_n(t,0)))$$

and $\pi_y F_{t,0}^{N+n}(D_n(t,0)) = D_y(0,2)$, by Lemma 3.0.5, we can see that

$$\frac{2}{C_1}|\mu|^{-n} \leqslant \operatorname{diam}(\pi_y F_{t,0}^N(D_n(t,0))) \leqslant 4C_1|\mu|^{-n}.$$

Then since $\operatorname{diam}(D_0(t,0)) > m > 0$, and $(D_y(0,2), D_0(t,0), \pi_y F_{t,0}^N)$ is a quadratic-like map, there is a uniform constant $C_2 > 0$ such that

$$\frac{1}{C_2|\mu|^{\frac{n}{2}}} \leqslant \operatorname{diam}(D_n(t,0)) \leqslant \frac{C_2}{|\mu|^{\frac{n}{2}}}.$$

Since $q_3(t) \in D_0(t,0)$ for every $n$ and the diameter of $D_0(t,0)$ is asymptotic to zero exponentially fast. We have

$$q_3(t) = \bigcap_n D_n(t,0).$$

$\square$

**Step 2.** Now we extend previous result from $(t,0)$ to $(t,a)$. Since by condition (5) in Section 2.5, we know

$$(D_y, W^u_{loc,q_3(t,a)}(p), \pi_y F_{t,a}^N)$$

is a quadratic-like map with unique critical point denoted by $q_3(t,a) = F_{t,a}^{-N}(q_1(t,a))$ and corresponding critical value $a$. By change of coordinates in $a-$parameter, there exist $A_0 = \mathbb{D}_a(0,2)$ such that for every $a \in A_0$, there exists a simply connected domain $D_0(t,a) \subset D_y$ such that

$$(D_y(0,2), D_0(t,a), \pi_y F_{t,a}^N)$$

is a quadratic-like map with critical point $q_3(t,a)$ and critical value $a \in D_y(0,2)$, and

$$dist(D_0(t,a), \mathbb{C} - D_y(0,2)) > 5\epsilon.$$

.

Now we will prove the following lemma to show similar construction exists for every $n$:

**Lemma 4.0.4.** *For every $t$, there exist a sequence of domains in the $a$-parameter, $A_n \subseteq \mathbb{D}_a(r)$ with $A_0 = \mathbb{D}_a(0,2)$ and $A_{n+1} \subseteq A_n$. For each $a \in A_n$, there is a simply connected domain*

*$D_n(t,a) \subseteq D_y$, such that*

$$(D_y(0,2), D_n(t,a), \pi_y F_{t,a}^{N+n})$$

*is a quadratic-like map with critical point denoted by $q_3(n,t,a)$ and critical value $v(n,t,a)$. Furthermore, we have an univalent onto mapping*

$$h_{t,n} : A_n \longrightarrow D_y(0,2)$$

*defined by $h_{t,n}(a) = v(n,t,a)$. $A_n$ is a simply-connected domain.*

*Moreover, their exist a uniform constant $C_3$ such that diameter of $A_n$ and $D_n(t,a)$ has the following estimation:*

$$\frac{1}{C_3|\mu|^n} \leqslant \mathrm{diam}(A_n) \leqslant \frac{C_3}{|\mu|^n}, \textit{ for } n \geqslant M,$$

$$\frac{1}{C_3|\mu|^{\frac{n}{2}}} \leqslant \mathrm{diam}(D_n(t,a)) \leqslant \frac{C_3}{|\mu|^{\frac{n}{2}}},$$

$$D_n(t,a) \subset D_y(q_3(n,t,0), \frac{C_3}{|\mu|^{\frac{n}{2}}}).$$

*Proof.* First, following the method used in the proof of Lemma 4.0.3, we construct $A_n$, $D_n(t,a)$ and $h_{t,n}$ inductively in n. $n = 0$ case is verified in the above discussion.

Suppose we have constructed $A_k$, $D_k(t,a), a \in A_k$ and the univalent mapping $h_{t,k}$ from $A_k$ to $D_y(0,2)$ for $k \leqslant n$. Then for every point $(0,z) \in D_n(t,a)$, where $a \in A_n$,

$$\begin{aligned}
|\frac{\pi_y F_{t,a}^{N+n+1}(0,z)}{\mu} - \pi_y F_{t,a}^{N+n}(0,z)| &= |(\frac{\pi_y F_{t,a}}{\mu} - \pi_y)(F_{t,a}^{N+n}(0,z))| \\
&= |E_n(F_{t,a}^{N+n}(0,z))\pi_y(F_{t,a}^{N+n}(0,z))| \\
&\leqslant \epsilon|\pi_y(F_{t,a}^{N+n}(0,z))| \\
&\leqslant 2\epsilon \\
&< \mathrm{d}(D_0(t,a), \mathbb{C} - D_y(0,2)) \\
&< \mathrm{d}(D_n(t,a), \mathbb{C} - D_y(0,2)).
\end{aligned}$$

Thus by Proposition 2.2.4, we know

$$(D_y(0,2-2\epsilon), (\frac{\pi_y F_{t,a}^{N+n+1}}{\mu})^{-1}(D_y(0,2-2\epsilon)), \frac{\pi_y F_{t,a}^{N+n+1}}{\mu})$$

is a quadratic-like map, for $a \in h_{n,t}^{-1}(D_y(0, 2-2\epsilon))$. Denote the critical value by $\frac{h_{n+1,t}(a)}{\mu}$. Then by proposition 2.2.3, we know that the critical value of $\frac{\pi_y F_{t,a}^{N+n+1}}{\mu}$ and the critical value of $F_{t,a}^{N+n}$ differ by $4\epsilon$, i.e.

$$|\frac{h_{n+1,t}(a)}{\mu} - h_{n,t}(a)| < 4\epsilon \tag{4.0.2}$$

where $a \in h_{n,t}^{-1}(D_y(0, 2-2\epsilon))$. Then by proposition 2.2.3, we know that $\frac{h_{n+1,t}(a)}{\mu}$ is univalent from $(\frac{h_{n+1,t}(a)}{\mu})^{-1}(D_y(0, 2-6\epsilon))$ to $D_y(0, 2-6\epsilon)$.

Thus

$$(D_y(0, |\mu|(2-2\epsilon)), (\pi_y F_{t,a}^{N+n+1})^{-1}(D_y(0, |\mu|(2-2\epsilon))), \pi_y F_{t,a}^{N+n+1})$$

is a quadratic-like map, for $a \in h_{n,t}^{-1}(D_y(0, 2-2\epsilon))$. Denote the critical point by $q_3(n+1, t, a)$ and the critical value by $v(n+1, t, a)$. Then we know that

$$v(n+1, t, a) = \mu \frac{h_{n+1,t}(a)}{\mu} = h_{n+1,t}(a).$$

Thus $h_{n+1,t}(a) = 0$ if and only if $\frac{h_{n+1,t}(a)}{\mu} = 0$ if and only if $a = 0$. Thus by proposition 2.1.1, we know $h_{n+1,t}(a)$ is univalent from $(h_{n+1,t})^{-1}(D_y(0, (2-6\epsilon)\mu))$ onto its image. Furthermore, since we have $\mu_{min}(2-6\epsilon) > 2$, we know $D_y(0, 2)$ is in the interior of the image. Thus we may define a simply connected domain $A_{n+1}$ to be the preimage $(h_{n+1,t})^{-1}(D_y(0, 2))$.

In conclusion, for $a \in A_{n+1}$, we have a quadratic-like map

$$(D_y(0, 2), D_{n+1}(t, a), \pi_y F_{t,a}^{N+n+1})$$

with critical point $q_3(n+1, t, a)$ and critical value $v(n+1, t, a)$, where $D_{n+1}(t, a)$ is defined by $(\pi_y F_{t,a}^{N+n+1})^{-1}(D_y(0, 2))$. Beside we know $h_{n+1,t}(a)$ defined by $h_{n+1,t}(a) = v(n+1, t, a)$ is univalent from $A_{n+1}$ onto $D_y(0, 2)$. This finish the induction step.

The estimation on the diameter of the $D_n(t, a)$ follows the same proof in Lemma 4.0.3 and the compactness of $A_n$. Now we estimate the diameter of $A_n$. By above calculations, we find that there exists constant $C > 0$ such that

$$\frac{1}{C}|\mu|^n \leqslant |\frac{\partial h_{n,t}}{\partial a}(0)| \leqslant C|\mu|^n.$$

Then by theorem 2.1.2, we can see that there exists constant $C_3 > 0$ such that

$$\frac{1}{C_3|\mu|^n} \leqslant \operatorname{diam}(A_n) \leqslant \frac{C_3}{|\mu|^n}$$

which finish the proof. □

*Remark* 4.0.5. The same argument can be applied if we relax $D_y(0,2)$ to $D_y(0,2+\epsilon)$. That is:

There exist a sequence of domains in the $a$-parameter, $\widetilde{A}_n \subseteq \mathbb{D}_a(r)$ with $\widetilde{A}_0 = \mathbb{D}_a(0,2+\epsilon)$ and $\widetilde{A}_{n+1} \subseteq \widetilde{A}_n$. For each $a \in \widetilde{A}_n$, there is a simply connected domain $\widetilde{D}_n(t,a) \subseteq D_y$, such that

$$(D_y(0,2+\epsilon), \widetilde{D}_n(t,a), \pi_y F_{t,a}^{N+n})$$

is a quadratic-like map with critical point denoted by $q_3(n,t,a)$ and critical value $v(n,t,a)$. Furthermore, we have an univalent onto mapping

$$h_{t,n} : \widetilde{A}_n \longrightarrow D_y(0,2+\epsilon)$$

defined by $h_{t,n}(a) = v(n,t,a)$. $\widetilde{A}_n$ is simply-connected.

Moreover, their exist a uniform constant $C_3$ such that diameter of $\widetilde{A}_n$ and $\widetilde{D}_n(t,a)$ has the following estimation:

$$\frac{1}{C_3|\mu|^n} \leqslant \operatorname{diam}(\widetilde{A}_n) \leqslant \frac{C_3}{|\mu|^n}, \text{ for } n \geqslant M,$$

$$\frac{1}{C_3|\mu|^{\frac{n}{2}}} \leqslant \operatorname{diam}(\widetilde{D}_n(t,a)) \leqslant \frac{C_3}{|\mu|^{\frac{n}{2}}},$$

$$D_n(t,a) \subset D_y(q_3(n,t,0), \frac{C_3}{|\mu|^{\frac{n}{2}}}).$$

Furthermore, $A_n$ is a proper subset of $\widetilde{A}_n$. And for every $a \in A_n$, $D_n(t,a)$ is a proper subset of $\widetilde{D}_n(t,a)$.

**Step 3.** Let $S$ be a domain in $D_{\vec{x}}$, denote $\mathcal{V}(S)$ be the standard vertical foliations in $D$ for $x \in S$, i.e., $\mathcal{V}(S)$ consists of 1-dimensional submanifolds of the form $L_{\vec{x}_0} = \{(\vec{x}_0, y) | y \in D_y\}$ where $\vec{x}_0 \in S$ as leaves. For every $n > 0$, we consider the foliation $F_{t,a}^n(\mathcal{V}(S))$. We parametrize the leaves of this push-forward foliation by the $\vec{x}$-coordinate of its preimage, and denote the leaf with parameter $\vec{x}$ by $\widetilde{L}_{n,\vec{x},t,a}$, i.e., $\widetilde{L}_{n,\vec{x},t,a} = F^n_{t,a}(L_{\vec{x}}) \cap D$. Thus by lemma 3.0.3, $\widetilde{L}_{n,\vec{x},t,a} \in \mathfrak{V}(\alpha)$. When $S$ is the polydisc in $D_{\vec{x}}$ centered at $\vec{0}$ with radius $r$, we simply write $\mathcal{V}(S)$ by $\mathcal{V}(r)$.

Now consider $\mathcal{V}_{n,t,a}(D) := F^n_{t,a}(\mathcal{V}(D_{\vec{x}})) \cap (D_{\vec{x}} \times \widetilde{D}_n(t,a))$, and denote the leaves of $\mathcal{V}_{n,t,a}(D)$ coming from $\widetilde{L}_{n,\vec{x},t,a}$ by $L_{n,\vec{x},t,a}$, i.e., $L_{n,\vec{x},t,a} = \widetilde{L}_{n,\vec{x},t,a} \cap (D_{\vec{x}} \times \widetilde{D}_n(t,a))$. Then $\pi_y|_{L_{n,\vec{x},t,a}}$ : $L_{n,\vec{x},t,a} \longrightarrow \widetilde{D}_n(t,a)$ gives a bi-holomorphism from $L_{n,\vec{x},t,a}$ to $\widetilde{D}_n(t,a)$. Thus we may define its inverse mapping, denoted by $\mathrm{Hol}_{n,t,a,\vec{x}}$, i.e., $\mathrm{Hol}_{n,t,a,\vec{x}} = (\pi_y|_{L_{n,\vec{x},t,a}})^{-1} : \widetilde{D}_n(t,a) \longrightarrow L_{n,\vec{x},t,a}$.

Now we consider the following mapping:

$$
\begin{aligned}
G(n,\vec{x},t,a) : \widetilde{D}_n(t,a) \longrightarrow & \qquad \mathbb{C} \\
y \longmapsto & \qquad \pi_y F^{N+n}_{t,a}(\mathrm{Hol}_{n,t,a,\vec{x}}(y))
\end{aligned}
$$

where $\vec{x} \in D_{\vec{x}}, a \in \widetilde{A}_n$.

We now prove the following lemma:

**Lemma 4.0.6.** *There exists a constant number $M > 0$ depending on $\epsilon$, such that for every $n > M$, we have the following:*

*For every $\vec{x} \in D_{\vec{x}}$, there exists a simply-connected domain $A_n(\vec{x})$ properly inside $\widetilde{A}_n$, such that for every $a \in A_n(\vec{x})$, there exists a simply-connected domain $D_n(t,a,\vec{x})$ properly inside $\widetilde{D}_n(t,a)$, such that*

$$(D_y(0,2), D_n(t,a,\vec{x}), G(n,\vec{x},t,a))$$

*is a quadratic-like map with critical point $q_3(n,t,a,\vec{x})$ and critical value $v(n,t,a,\vec{x})$. Furthermore, the mapping*

$$h_{n,t,\vec{x}} : A_n(\vec{x}) \longrightarrow D_y(0,2)$$

*defined by $h_{n,t,\vec{x}}(a) := v(n,t,a,\vec{x})$ is an univalent mapping onto its image. And their exist an uniform constant $C_4 \geqslant C_3$ such that diameter of $A_n(\vec{x})$ and $D_n(t,a,\vec{x})$ has the following estimation:*

$$\frac{1}{C_4|\mu|^n} \leqslant \mathrm{diam}(A_n(\vec{x})) \leqslant \frac{C_4}{|\mu|^n}, \ \textit{for } n \geqslant M,$$

$$\frac{1}{C_4|\mu|^{\frac{n}{2}}} \leqslant \operatorname{diam}(D_n(t,a,\vec{x})) \leqslant \frac{C_4}{|\mu|^{\frac{n}{2}}}.$$

$$D_y(0, 2-\frac{1}{3}\epsilon) \subseteq h_{n,t}(A_{n,t}(\vec{x})) \subseteq D_y(0, 2+\frac{1}{3}\epsilon).$$

*where* Mod *denotes the modulus of the annulus.*

*Remark* 4.0.7. When $\vec{x} = \vec{0}$, we have

$$A_n(\vec{0}) = A_n \text{ and } D_n(t,a,\vec{0}) = D_n(t,a).$$

Thus lemma 4.0.4 is just the spacial case of lemma 4.0.6 for $\vec{x} = \vec{0}$.

*Proof.* Let $(\vec{0}, y)$ be a point in $\widetilde{D}_n(t,a)$. Since $\mathrm{Hol}_{n,t,a,\vec{x}}(\widetilde{D}_n(t,a))$ is contained in $F^n_{t,a}(L_{\vec{x}})$, by lemma 3.0.5, we know

$$|\,\mathrm{Hol}_{n,t,a,\vec{x}}(y) - (\vec{0}, y)| \leqslant C_1|\vec{x}||\lambda|^n.$$

Thus

$$|\pi_y F^N_{t,a}(\mathrm{Hol}_{n,t,a,\vec{x}}(y)) - \pi_y F^N_{t,a}((\vec{0}, y))| = O(|\lambda|^n).$$

Furthermore, by lemma 3.0.5, we have

$$|\pi_y F^{N+n}_{t,a}(\mathrm{Hol}_{n,t,a,\vec{x}}(y)) - \pi_y F^{N+n}_{t,a}((\vec{0}, y))| = O(|\lambda\mu|^n). \tag{4.0.3}$$

Thus we know there exists a constant $M_1 > 0$ such that for every $n > M_1$,

$$|\pi_y F^{N+n}_{t,a}(\mathrm{Hol}_{n,t,a,\vec{x}}(y)) - \pi_y F^{N+n}_{t,a}((\vec{0}, y))| \leqslant \frac{1}{3}\epsilon. \tag{4.0.4}$$

Then by proposition 2.2.4, for every $a \in h^{-1}_{n,t}(D_y(0, 2+\frac{2}{3}\epsilon))$,

$$\left(D_y(0, 2+\frac{2}{3}\epsilon), (G(n,\vec{x},t,a))^{-1}(D_y(0, 2+\frac{2}{3}\epsilon)), G(n,\vec{x},t,a)\right) \tag{4.0.5}$$

is a quadratic-like map, holomorphically depends on $\vec{x}, t, a$. Define $D_n(t,a,\vec{x})$ by

$$D_n(t,a,\vec{x}) := (G(n,\vec{x},t,a))^{-1}(D_y(0,2)).$$

Since $D_y(0,2)$ is proper subset of $D_y(0, 2+\frac{1}{3}\epsilon)$, we may define $A_n(\vec{x}) \subsetneqq h^{-1}_{n,t}(D_y(0, 2+\frac{1}{3}\epsilon))$ by

$$A_{n,t}(\vec{x}) := \{a \in \widetilde{A}_n | (D_y(0,2), D_n(t,a,\vec{x}), G(n,\vec{x},t,a))$$

is a quadratic-like map with critical value contained in $D_y(0,2)\}$, and $A_{n,t}(\vec{x})$ is a simply-connected domain properly inside $\widetilde{A}_n$. Then we can define the following holomorphic mapping:

$$h_{n,t,\vec{x}} : A_{n,t}(\vec{x}) \longrightarrow D_y(0,2)$$

by

$$h_{n,t,\vec{x}}(a) := v(n,t,a,\vec{x}).$$

Denote $i_D$ be the restriction onto $D$. Then by Inclination Lemma, since $L_{\vec{x}}$ intersects with stable manifold transversally, we know that

$$\lim_{n\to\infty}(i_D \circ F_{t,a})^n L_{\vec{x}} = D_y$$

in $C^1$-topology. Thus for any given $\vec{x}$, $h_{n,t,\vec{x}}$ converge to $h_{n,t}$ as $n$ goes to $\infty$ in $C^1$-topology. Then by Theorem 2.5 in [PP11], we conclude that $h_{n,t,\vec{x}}$ is univalent when $n$ large enough.

We finish the proof by noticing that the estimations on the diameters of $A_n(\vec{x})$ and $D_n(t,a,\vec{x})$ follows the same argument as in lemma 4.0.4.

Also when $h_{n,t,\vec{x}}$ are univalent for all $\vec{x} \in D_{\vec{x}}$, we have the following estimation:

$$D_y(0, 2 - \frac{1}{3}\epsilon) \subseteq h_{n,t}(A_{n,t}(\vec{x})) \subseteq D_y(0, 2 + \frac{1}{3}\epsilon).$$

□

**Lemma 4.0.8.** *When $n$ large enough, there exists a holomorphic mapping $\mathrm{sa}_{n,t} : D_{\vec{x}} \longrightarrow \mathbb{C}$ such that the following holds:*

$$q_3(n,t,a,\vec{x}) = v(n,t,a,\vec{x}) \text{ if and only if } a = \mathrm{sa}_{n,t}(\vec{x}).$$

*Proof.* By lemma 4.0.6, when $n$ large enough, $h_{n,t,\vec{x}}$ is univalent. Thus we may define the following mapping:

$$P_{n,t,\vec{x}} : D_y(0,2) \longrightarrow \mathbb{C}$$

by

$$P_{n,t,\vec{x}}(z) := q_3(n,t,(h_{n,t,\vec{x}})^{-1}(z),\vec{x}).$$

Then it is a holomorphic mapping by the definition of $h_{n,t,\vec{x}}$ and $q_3(n,t,a,\vec{x})$. Since we have $q_3(n,t,a,\vec{x}) \in \widetilde{D}_0(t,a) \subsetneqq D_y(0,2)$ for $a \in A_{n,t}(\vec{x})$, we know that $P_{n,t,\vec{x}}(D_y(0,2))$ lies strictly inside $D_y(0,2)$. By Earle-Hamilton holomorphic fixed point theorem ( Theorem 2.3.1), there exists a unique fixed point $v$ of $P_{n,t,\vec{x}}$ in $D_y(0,2)$.

Now we define $\mathrm{sa}_{n,t}(\vec{x})$ to be $h^{-1}_{n,t,\vec{x}}(v)$, then it is easy to see it is a holomorphic mapping and holomorphically depending on $t$. By the uniqueness of the fixed point, we can see that

$$q_3(n,t,a,\vec{x}) = v(n,t,a,\vec{x}) \text{ if and only if } a = \mathrm{sa}_{n,t}(\vec{x})$$

which finish the proof. $\square$

Now since

$$(D_y(0,2), D_n(t,a,\vec{x}), G(n,\vec{x},t,a))$$

is a quadratic-like map with critical point $q_3(n,t,a,\vec{x})$ and critical value $v(n,t,a,\vec{x})$, we may denote the following:

$$\mathrm{Hol}_{n,t,a,\vec{x}}(q_3(n,t,a,\vec{x})) := (X_1(n,t,a,\vec{x}), q_3(n,t,a,\vec{x}))$$

and

$$F^{N+n}_{t,a}(X_1(n,t,a,\vec{x}), q_3(n,t,a,\vec{x})) := (X_2(n,t,a,\vec{x}), v(n,t,a,\vec{x})).$$

**Lemma 4.0.9.** *For $n$ large enough, there exists a point $x^*_{n,t} \in D_{\vec{x}}$, such that the following holds:*

$$X_1(n,t,\mathrm{sa}_{n,t}(\vec{x}),\vec{x}) = X_2(n,t,\mathrm{sa}_{n,t}(\vec{x}),\vec{x}) \textit{ if and only if } \vec{x} = x^*_{n,t}.$$

*Proof.* By lemma 4.0.8, we know that $X_i(n,t,\mathrm{sa}_{n,t}(\vec{x}),\vec{x})$ holomorphically depending on $\vec{x}$, where $i = 1,2$. Besides, we have

$$\pi_{\vec{x}} F^{-n}_{t,\mathrm{sa}_{n,t}(\vec{x})}(X_1(n,t,\mathrm{sa}_{n,t}(\vec{x}),\vec{x}), q_3(n,t,\mathrm{sa}_{n,t}(\vec{x}),\vec{x})) = \vec{x}.$$

Now we may define the a holomorphic mapping

$$S_{n,t} : D_{\vec{x}} \longrightarrow \mathbb{C}^{m-1}$$

by the following:

$$S_{n,t}(\vec{x}) = \pi_{\vec{x}} F^N_{t,\mathrm{sa}_{n,t}(\vec{x})}(X_1(n,t,\mathrm{sa}_{n,t}(\vec{x}),\vec{x}), q_3(n,t,\mathrm{sa}_{n,t}(\vec{x}),\vec{x})).$$

Since $|X_1(n,t,\mathrm{sa}_{n,t}(\vec{x}))| = O(|\lambda|^n)$, when $n$ large enough, we know $S_{n,t}(\vec{x}) \in D_{\vec{x}}(2)$, where $D_{\vec{x}}(2) = \{(\vec{x},0) \mid |x_i| \leqslant 2, 1 \leqslant i \leqslant m-1\}$.

Since $L > 2$, we know that $S_{n,t}(D_{\vec{x}})$ is strictly inside $D_{\vec{x}}$. By Earle-Hamilton holomorphic fixed point theorem ( Theorem 2.3.1), there exists a unique fixed point of $S_{n,t}(\vec{x})$ in $D_{\vec{x}}$. We denote that fixed point by $x^*_{n,t}$.

Since

$$\pi_{\vec{x}} F^N_{t,\mathrm{sa}_{n,t}(\vec{x})}(X_1(n,t,\mathrm{sa}_{n,t}(\vec{x}),\vec{x}), q_3(n,t,\mathrm{sa}_{n,t}(\vec{x}),\vec{x}))$$

can be rewritten as

$$\pi_{\vec{x}} F^{-n}_{t,\mathrm{sa}_{n,t}(\vec{x})}(X_2(n,t,\mathrm{sa}_{n,t}(\vec{x}),\vec{x}), v(n,t,\mathrm{sa}_{n,t}(\vec{x}),\vec{x})),$$

we know, when $\vec{x} = x^*_{n,t}$,

$$F^{-n}_{t,\mathrm{sa}_{n,t}(x^*_{n,t})}(X_1(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t}), q_3(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t}))$$

and

$$F^{-n}_{t,\mathrm{sa}_{n,t}(x^*_{n,t})}(X_2(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t}), v(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t}))$$

are both in $L_{x^*_{n,t}}$. Thus

$$(X_1(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t}), q_3(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t}))$$

and

$$(X_2(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t}), v(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t}))$$

are both in $F^n_{t,\mathrm{sa}_{n,t}(x^*_{n,t})}(L_{x^*_{n,t}}) \cap D$. Since $F^n_{t,\mathrm{sa}_{n,t}(x^*_{n,t})}(L_{x^*_{n,t}}) \cap D$ is almost vertical, it can be represented as the graph over $D_y$, but we know that

$$q_3(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t}) = v(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t}),$$

we must have

$$X_1(n, t, \mathrm{sa}_{n,t}(x^*_{n,t}), x^*_{n,t}) = X_2(n, t, \mathrm{sa}_{n,t}(x^*_{n,t}), x^*_{n,t}).$$

Since $x^*_{n,t}$ is the unique fixed point of $S_{n,t}$, we can see that

$$q_3(n, t, a, \vec{x}) = v(n, t, a, \vec{x}) \text{ if and only if } a = \mathrm{sa}_{n,t}(\vec{x})$$

Which finish the proof. $\square$

*Remark* 4.0.10. We actually know that

$$(X_1(n, t, a, \vec{x}), q_3(n, t, a, \vec{x}))$$

and

$$(X_2(n, t, a, \vec{x}), v(n, t, a, \vec{x})) = F^{N+n}_{t,a}(X_1(n, t, a, \vec{x}), q_3(n, t, a, \vec{x}))$$

coincide when $\vec{x} = x^*_{n,t}$ and $a = \mathrm{sa}_{n,t}(x^*_{n,t})$. Thus

$$(X_1(n, t, \mathrm{sa}_{n,t}(x^*_{n,t}), x^*_{n,t}), q_3(n, t, \mathrm{sa}_{n,t}(x^*_{n,t}), x^*_{n,t}))$$

is a periodic point of $F_{t,\mathrm{sa}_{n,t}(x^*_{n,t})}$ with period $N + n$.

Next, we prove that the periodic point is a sink. We know

$$(D_y(0, 2), D_n(t, \mathrm{sa}_{n,t}(x^*_{n,t}), x^*_{n,t}), G(n, x^*_{n,t}, t, \mathrm{sa}_{n,t}(x^*_{n,t})))$$

is a quadratic-like map with a superattracting critical fixed point $q_3(n, t, \mathrm{sa}_{n,t}(x^*_{n,t}), x^*_{n,t})$. We may choose 2 univalent mappings

$$\phi_i(n, t) : D_y \longrightarrow \mathbb{C}_z$$

where $i = 1, 2$ and $\mathbb{C}_z$ denotes the complex plane with coordinate $z$, such that the following holds:

1) $\phi_1(n, t)$ mappings $D_n(t, \mathrm{sa}_{n,t}(x^*_{n,t}), x^*_{n,t})$ biholomorphically onto $D_z(0, 1)$, with $q_3(n, t, \mathrm{sa}_{n,t}(x^*_{n,t}), x^*_{n,t})$ to 0;

2) $\phi_2(n,t)$ mappings $D_y(0,2)$ biholomorphically onto $D_z(0,1)$,with $q_3(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t})$ to 0, we choose $\phi_2(n,t)$ to have the explicit formula:

$$\phi_2(n,t)(y) = \frac{\frac{1}{2}y - \frac{1}{2}q_3(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t})}{1 - \frac{1}{4}y\overline{q_3(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t})}}$$

It is just the composition of the linear mapping $y \longrightarrow \frac{1}{2}y$ and the Möbius mapping

$$z = \frac{y - \frac{1}{2}q_3(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t})}{1 - \frac{1}{2}y\overline{q_3(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t})}}.$$

Thus, $\phi_2(n,t) \circ G(n,x^*_{n,t},t,\mathrm{sa}_{n,t}(x^*_{n,t})) \circ (\phi_1(n,t))^{-1}$ is a degree 2 mapping on the unit disc $D_z(0,1)$ with unique superattrating fixed point 0. Thus by complex analysis, we see that

$$\phi_2(n,t) \circ G(n,x^*_{n,t},t,\mathrm{sa}_{n,t}(x^*_{n,t})) \circ (\phi_1(n,t))^{-1}(z) = \xi z^2,$$

where $\xi$ is a complex number with $|\xi| = 1$. Thus we have

$$G(n,x^*_{n,t},t,\mathrm{sa}_{n,t}(x^*_{n,t}))(y) = \phi_2(n,t)^{-1}(\xi(\phi_1(n,t)(y))^2). \tag{4.0.6}$$

Now we prove the following lemma:

**Lemma 4.0.11.** *There exists a constant $\tau = C(\epsilon) > 0$ only depending on $\epsilon$, independent of $n$ and $t$, such that , if we denote*

$$E_{n,t}(\alpha) := D_y(q_3(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t}), \frac{\tau\alpha}{|\mu|^n}),$$

*then we have the following:*

*1)*

$$E_{n,t}(1) \subsetneq D_n(t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t}),$$

*2)*

$$(G(n,x^*_{n,t},t,\mathrm{sa}_{n,t}(x^*_{n,t})))E_{n,t}(1) \subseteq E_{n,t}(\frac{1}{2}) \text{ when } n \text{ large enough.}$$

*Proof.* For simplicity of writing, we will denote $D_n(t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t})$, $G(n,x^*_{n,t},t,\mathrm{sa}_{n,t}(x^*_{n,t}))$, $\phi_i(n,t)$, $q_3(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t})$ and $\mu$ by $D_n$, $G$, $\phi_i$, $q_3$ and $\mu$ respectively in this proof, where $i = 1,2$. Thus by equation (4.0.6) is rewritten as

$$G(y) = \phi_2^{-1}(\xi(\phi_1(y))^2),$$

where $\phi_2$ simply rewrite as

$$\phi_2(y) = \frac{\frac{1}{2}y - \frac{1}{2}q_3}{1 - \frac{1}{4}y\overline{q_3}}.$$

Consider a disc $D_z(0, \frac{\tau'}{|\mu|^{\frac{n}{2}}})$ in the $z$-plane with center 0 and radius $\frac{\tau'}{|\mu|^{\frac{n}{2}}}$, where $\tau' > 0$ with be determined later. Then we have

$$(\xi\phi_1^2)(D_z(0, \frac{\tau'}{|\mu|^{\frac{n}{2}}})) = D_z(0, \frac{(\tau')^2}{|\mu|^n}).$$

By Koebe $\frac{1}{4}$ Theorem, we have the following:

$$\begin{aligned} \phi_1^{-1}(D_z(0, \frac{\tau'}{|\mu|^{\frac{n}{2}}})) &\supseteq D_y(q_3, \frac{1}{4}|(\phi_1^{-1})'(0)|\frac{\tau'}{|\mu|^{\frac{n}{2}}})), \\ \phi_2^{-1}(D_z(0, \frac{(\tau')^2}{|\mu|^n}) &\subseteq D_y(q_3, 4|(\phi_2^{-1})'(0)|\frac{(\tau')^2}{|\mu|^n})). \end{aligned}$$

Thus we have

$$\begin{aligned} G(D_y(q_3, \frac{1}{4}|(\phi_1^{-1})'(0)|\frac{\tau'}{|\mu|^{\frac{n}{2}}})) &= (\phi_2^{-1} \circ \xi z^2 \circ \phi_1)(D_y(q_3, \frac{1}{4}|(\phi_1^{-1})'(0)|\frac{\tau'}{|\mu|^{\frac{n}{2}}})) \\ &\subseteq (\phi_2^{-1} \circ \xi z^2)(D_z(0, \frac{\tau'}{|\mu|^{\frac{n}{2}}})) \\ &= (\phi_2^{-1})(D_z(0, \frac{(\tau')^2}{|\mu|^n})) \\ &\subseteq D_y(q_3, 4|(\phi_2^{-1})'(0)|\frac{(\tau')^2}{|\mu|^n})). \end{aligned} \tag{4.0.7}$$

Let $\tau_1 = \frac{1}{4}|(\phi_1^{-1})'(0)|\frac{\tau'}{|\mu|^{\frac{n}{2}}}|\mu|^n = \frac{1}{4}|(\phi_1^{-1})'(0)|\tau'|\mu|^{\frac{n}{2}}$, then formula (4.0.7) is equivalent to the following:

$$G(D_y(q_3, \frac{\tau_1}{|\mu|^n})) \subseteq D_y(q_3, \frac{64|(\phi_2^{-1})'(0)|\tau_1^2}{|\mu|^{2n}|(\phi_1^{-1})'(0)|^2})$$

The lemma will be proved if the following condition is satisfied for $\tau_1 = \tau$:

$$D_y(q_3, \frac{64|(\phi_2^{-1})'(0)|\tau_1^2}{|\mu|^{2n}|(\phi_1^{-1})'(0)|^2}) \subseteq D_y(q_3, \frac{\tau_1}{2|\mu|^n}). \tag{4.0.8}$$

It is equivalent to the following inequality:

$$\frac{64|(\phi_2^{-1})'(0)|\tau^2}{|\mu|^{2n}|(\phi_1^{-1})'(0)|^2} \leqslant \frac{\tau}{2|\mu|^n}. \tag{4.0.9}$$

Since $(\phi_2^{-1})'(0) = \frac{1}{2} - \frac{1}{8}|q_3|^2$, thus it is equivalent to the following inequality:

$$\tau \leqslant \frac{|\mu|^n|(\phi_1^{-1})'(0)|^2}{16(4 - |q_3|^2)}. \tag{4.0.10}$$

Now we give a lower bound estimation of $|(\phi_1^{-1})'(0)|$.

Since $q_3 \in \widetilde{D}_0(t) \subset D_y(0, 2-\epsilon)$, we have

$$\mathrm{sa}_{n,t}(x^*_{n,t}) \in h^{-1}_{n,t,x^*_{n,t}}(D_y(0, 2-\epsilon)),$$

and $(D_y(0, 2-\epsilon), G^{-1}(D_y(0, 2-\epsilon)), G)$ is also a quadratic-like map. By the same method as in the proof of Lemma 4.0.6, we may have the following estimation on the diameter of $G^{-1}(D_y(0, 2-\epsilon))$:

$$\frac{1}{C_4(\epsilon)|\mu|^{\frac{n}{2}}} \leqslant \mathrm{diam}(G^{-1}(D_y(0, 2-\epsilon))) \leqslant \frac{C_4(\epsilon)}{|\mu|^{\frac{n}{2}}},$$

where $C_4(\epsilon)$ is a uniform constant only depend on $C_4$ and $\epsilon$. Besides, by Lemma 2.2.5, we have

$$\mathrm{Mod}(D_n \setminus G^{-1}(D_y(0, 2-\epsilon))) \geqslant \frac{1}{2}\mathrm{Mod}(D_y(0, 2) \setminus D_y(0, 2-\epsilon)) = \frac{1}{4\pi}\log(\frac{2}{2-\epsilon}) > 0.$$

Since $\phi_1$ is univalent on $D_n$, by Theorem 2.1.2, we have a constant $C > 0$ depending on $\frac{1}{4\pi}\log(\frac{2}{2-\epsilon})$, such that for any $x, y, z \in G^{-1}(D_y(0, 2-\epsilon))$,

$$\frac{1}{C}\left|\phi_1'(x)\right| \leqslant \frac{|\phi_1(y) - \phi_1(z)|}{|y - z|} \leqslant C\left|\phi_1'(x)\right|.$$

By choosing sequences of points $y_i, z_i \in \mathrm{diam}(G^{-1}(D_y(0, 2-\epsilon))$ such that $\lim |y_i - z_i| = \mathrm{diam}(G^{-1}(D_y(0, 2-\epsilon))$ and using the fact that $|\phi_1(y) - \phi_1(z)| \leqslant 2$ for any $y, z \in G^{-1}(D_y(0, 2-\epsilon)$, we have

$$|\phi_1'(q_3)| \leqslant C\frac{2}{\mathrm{diam}(G^{-1}(D_y(0, 2-\epsilon))} \leqslant 2C_4(\epsilon)C|\mu|^{\frac{n}{2}}$$

Since $\phi_1'(q_3) = \frac{1}{(\phi_1^{-1})'(0)}$, we have

$$|(\phi_1^{-1})'(0)| \geqslant \frac{1}{2C_4(\epsilon)C|\mu|^{\frac{n}{2}}}$$

Thus for $\tau = \frac{1}{256C_4(\epsilon)^2C^2}$, we have

$$\frac{|\mu|^n|(\phi_1^{-1})'(0)|^2}{16(4 - |q_3|^2)} \geqslant \frac{1}{64C_4(\epsilon)^2C^2(4 - |q_3|^2)} > \tau$$

Thus $\tau_1 = \tau$ satisfies inequality (4.0.10). Thus we have

$$G(D_y(q_3, \frac{\tau}{|\mu|^n})) \subseteq D_y(q_3, \frac{64|(\phi_2^{-1})'(0)|\tau^2}{|\mu|^{2n}|(\phi_1^{-1})'(0)|^2}) \subseteq D_y(q_3, \frac{\tau}{2|\mu|^n}).$$

Let $C(\epsilon) = \frac{1}{256C_4(\epsilon)^2C^2}$, we can see that it only depends on $\epsilon$, independent of $n$ and $t$. This finishes the proof. $\square$

Since $q_3(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}), x^*_{n,t}) \in D_n(t,a,x^*_{n,t}) \subset D_y(0, 2-\epsilon)$ and by equation ( 4.0.4), we have

$$|\pi_y F^{N+n}_{t,a}(\mathrm{Hol}_{n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),\vec{x}}(y)) - \pi_y F^{N+n}_{t,a}(\mathrm{Hol}_{n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t}}(y))| \leqslant \frac{2}{3}\epsilon$$

for every $y \in \widetilde{D}_n(t,a)$ and every $\vec{x} \in D_{\vec{x}}$. By Proposition 2.2.4, we conclude that

$$(D_y(0,2), D_n(t,a,\vec{x}), G(n,\vec{x},t,\mathrm{sa}_{n,t}(x^*_{n,t})))$$

is quadratic-like for every $\vec{x} \in D_{\vec{x}}$, i.e.,

$$\mathrm{sa}_{n,t}(x^*_{n,t}) \in A_{n,t}(\vec{x}).$$

Now we prove the following lemma:

**Lemma 4.0.12.** *Let $H_{n,t}(\kappa)$ be a polydisc in the $\vec{x}$-plane centered at $\vec{0}$ with radius $\kappa|\lambda|^n$. Then there exist a constant $\kappa > 4C_1$ independent of $n$ and $t$ such that*

$$F^{n+N}_{t,\mathrm{sa}_{n,t}(x^*_{n,t})}(H_{n,t}(\kappa) \times E_{n,t}(1)) \subseteq (H_{n,t}(\frac{1}{2}\kappa) \times E_{n,t}(\frac{3}{4})) \text{ when } n \text{ large enough.}$$

*It then implies that*

$$(X_1(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}), x^*_{n,t}), q_3(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}), x^*_{n,t}))$$

*is a sink.*

*Proof.* By equation (4.0.3), we have

$$\begin{aligned}
&|\pi_y F^{N+n}_{t,\mathrm{sa}_{n,t}(x^*_{n,t})}(\mathrm{Hol}_{n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),\vec{x}}(y)) - \pi_y F^{N+n}_{t,\mathrm{sa}_{n,t}(x^*_{n,t})}(\mathrm{Hol}_{n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t}}(y))| \\
&\leqslant |\pi_y F^{N+n}_{t,\mathrm{sa}_{n,t}(x^*_{n,t})}(\mathrm{Hol}_{n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),\vec{x}}(y)) - \pi_y F^{N+n}_{t,a}((\vec{0},y))| + \\
&|\pi_y F^{N+n}_{t,a}((\vec{0},y)) - \pi_y F^{N+n}_{t,\mathrm{sa}_{n,t}(x^*_{n,t})}(\mathrm{Hol}_{n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t}}(y))| \\
&= O(|\lambda\mu|^n)
\end{aligned}$$

for every $\vec{x} \in D_{\vec{x}}$ and $y \in \widetilde{D}_n(t,a)$. By condition (F3) in Definition 2.4.4, for $n$ large enough, we have

$$\begin{aligned}
&|\pi_y F^{N+n}_{t,\mathrm{sa}_{n,t}(x^*_{n,t})}(\mathrm{Hol}_{n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),\vec{x}}(y)) - \pi_y F^{N+n}_{t,\mathrm{sa}_{n,t}(x^*_{n,t})}(\mathrm{Hol}_{n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),x^*_{n,t}}(y))| \\
&= O(\frac{1}{|\mu|^{2n}}) \\
&\leqslant \frac{\tau}{4|\mu|^n}
\end{aligned}$$

Thus we have

$$G(n,t,\mathrm{sa}_{n,t}(x^*_{n,t}),\vec{x})(E_{n,t}(1)) \subseteq E_{n,t}(\frac{3}{4})$$

Besides, by the normalization of the problem, when $n$ large enough we can see that

$$\pi_{\vec{x}} F^N_{t,\mathrm{sa}_{n,t}(x^*_{n,t})}(H_{n,t}(\kappa) \times E_{n,t}(1)) \subseteq \pi_x D_2.$$

Thus by lemma 3.0.5 we have

$$\pi_{\vec{x}} F^{N+n}_{t,\mathrm{sa}_{n,t}(x^*_{n,t})}(H_{n,t}(\kappa) \times E_{n,t}(1)) \subseteq H_{n,t}(2C_1).$$

Choose a constant $\kappa$ such that $\frac{1}{2}\kappa > 2C_1$. Then we have

$$F^{n+N}_{t,\mathrm{sa}_{n,t}(x^*_{n,t})}(H_{n,t}(\kappa) \times E_{n,t}(1)) \subseteq (H_{n,t}(\frac{1}{2}\kappa) \times E_{n,t}(\frac{3}{4}))$$

which finish the proof. □

**Step 4.** In this part we finish the proof of Theorem 4.0.1. Now choose $\eta > 0$ small enough, consider a disc around $\mathrm{sa}_{n,t}(x^*_{n,t})$ in $A_n(x^*_{n,t})$ with radius $\frac{\eta}{|\mu|^{2n}}$, denote it as $D_a(\mathrm{sa}_{n,t}(x^*_{n,t}), \frac{\eta}{|\mu|^{2n}})$.

Then for every $a \in D_a(\mathrm{sa}_{n,t}(x^*_{n,t}), \frac{\eta}{|\mu|^{2n}})$, by Taylor expansion of $F^N_{t,a}(\vec{x},y)$ on $(\vec{x},y,a)$ parameter, we know the following:

$$\begin{aligned}
&\pi_y F^N_{t,a}(H_{n,t}(\kappa) \times E_{n,t}(1)) \text{ is in } \frac{C\eta}{|\mu|^{2n}}\text{-neighbourhood of} \\
&\pi_y F^N_{t,\mathrm{sa}_{n,t}(x^*_{n,t})}(H_{n,t}(\kappa) \times E_{n,t}(1))
\end{aligned}$$

where $C > 0$ is some constant uniformly away from 0.

Besides by the lemma in Appendix, we know

$$\pi_y F^{-n}_{t,\mathrm{sa}_{n,t}(x^*_{n,t})}(H_{n,t}(\frac{1}{2}\kappa) \times E_{n,t}(\frac{3}{4})) \text{ is in } O(\frac{n|a - sa_n|}{|\mu|^n}) = O(\frac{n}{|\mu|^{3n}})$$
$$\text{-neighbourhood of } \pi_y F^{-n}_{t,a}(H_{n,t}(\frac{1}{2}\kappa) \times E_{n,t}(\frac{3}{4})).$$

Thus if we choose $\eta$ small enough such that $O(\frac{n}{|\mu|^{3n}}) + \frac{C\eta}{|\mu|^{2n}} \leqslant \frac{\tau}{8C_1|\mu|^{2n}}$. Then since

$$\pi_y F^{N}_{t,\mathrm{sa}_{n,t}(x^*_{n,t})}(H_{n,t}(\kappa) \times E_{n,t}(1)) \subseteq \pi_y F^{-n}_{t,\mathrm{sa}_{n,t}(x^*_{n,t})}(H_{n,t}(\frac{1}{2}\kappa) \times E_{n,t}(\frac{3}{4})),$$

we find

$$\pi_y F^{N}_{t,a}(H_{n,t}(\kappa) \times E_{n,t}(1)) \text{ is in } \frac{\tau}{8C_1|\mu|^{2n}}\text{-neighbourhood of}$$
$$\pi_y F^{-n}_{t,a}(H_{n,t}(\frac{1}{2}\kappa) \times E_{n,t}(\frac{3}{4})).$$

Thus by lemma 3.0.5, we have

$$\pi_y F^{N+n}_{t,a}(H_{n,t}(\kappa) \times E_{n,t}(1)) \text{ is in } \frac{\tau}{8|\mu|^{n}}\text{-neighbourhood of}$$
$$\pi_y(H_{n,t}(\frac{1}{2}\kappa) \times E_{n,t}(\frac{3}{4})) = E_{n,t}(\frac{3}{4}),$$

i.e.,

$$\pi_y F^{N+n}_{t,a}(H_{n,t}(\kappa) \times E_{n,t}(1)) \subseteq E_{n,t}(\frac{7}{8}).$$

Thus by lemma 3.0.5 we have

$$\pi_{\vec{x}} F^{N+n}_{t,a}(H_{n,t}(\kappa) \times E_{n,t}(1)) \subseteq H_{n,t}(2C_1) \subseteq H_{n,t}(\frac{3}{4}\kappa)$$

by the choice of $\kappa$. Overall, we have

$$F^{n+N}_{t,a}(H_{n,t}(\kappa) \times E_{n,t}(1)) \subseteq (H_{n,t}(\frac{3}{4}\kappa) \times E_{n,t}(\frac{7}{8})).$$

Thus there exists a unique sink of mapping $F^{n+N}_{t,a}$ inside $H_{n,t}(\kappa) \times E_{n,t}(1)$. Since $x^*_{n,t}$ is holomorphically depending on $t$, then if we define the mapping

$$a_n : \mathbb{D}_t(r) \longrightarrow \mathbb{D}_a(r)$$

by

$$a_n(t) := \mathrm{sa}_{n,t}(x^*_{n,t}),$$

we can see that $a_n(t)$ is a holomorphic function. This finish the proof of Proposition 4.0.1. $\square$

# Chapter 5

# The creation and properties of secondary tangency

## 5.1 The creation of secondary tangency

In this section, we will create a map $b_{n,k,i} : T_\mu(\mu_{min}, \mu_{max}) \times T_\lambda(\lambda_{min}, \lambda_{max}) \times U \longrightarrow \mathbb{D}_a(2)$ such that when parameter are on the graph of this map, $F^{N+\theta n+N+n+N+k}$ will exhibit the a new homoclinic tangency. We will show this result by construction in several steps, again, we fix a $t = (\mu, \lambda, \tau)$.

**Step 1.** First of all, we will consider iterations with iternery $N + \theta n$. We begin with the following lemma:

**Lemma 5.1.1.** *There exists a constant $\chi > 0$ such that, for $n > 2\chi$, we have*

$$h_{k,t}(A_n) \subseteq D_y(0,2) \backslash D_0(t,0)$$

*for every $k < n - \chi$. Especially, when $n > \frac{\chi}{1-\theta}$, we have*

$$\theta n < n - \chi,$$

*thus*

$$h_{\theta n,t}(A_n) \subseteq D_y(0,2) \backslash D_0(t,0).$$

*Proof.* We know that $D_y(0,1) \subset D_y(0,2) \backslash D_0(t,0)$. By inequality (4.0.2), for $a \in A_n$, we have

$$\begin{aligned}
|h_{k,t}(a) - \frac{h_{n,t}(a)}{\mu^{n-k}}| &= |\sum_{i=k}^{n-1}(\frac{h_{i,t}(a)}{\mu^{i-k}} - \frac{h_{i+1,t}(a)}{\mu^{i+1-k}})| \\
&\leqslant \sum_{i=k}^{n-1}|\frac{h_{i,t}(a)}{\mu^{i-k}} - \frac{h_{i+1,t}(a)}{\mu^{i+1-k}}| \\
&< 4\epsilon\sum_{i=k}^{n-1}|\frac{1}{\mu}|^{i-k} \\
&< 4\epsilon\sum_{i=0}^{\infty}|\frac{1}{\mu}|^{i} \\
&= \frac{4\epsilon|\mu|}{|\mu|-1}
\end{aligned}$$

Thus we have

$$|h_{k,t}(a)| < |\frac{h_{n,t}(a)}{\mu^{n-k}}| + \frac{4\epsilon|\mu|}{|\mu|-1} \leqslant \frac{2}{|\mu|^{n-k}} + \frac{4\epsilon|\mu|}{|\mu|-1} \leqslant \frac{2}{\mu_{min}^{n-k}} + \frac{4\epsilon\mu_{min}}{\mu_{min}-1}$$

Let

$$\chi = \frac{\ln(2-\frac{2}{\mu_{min}}) - \ln(1-\frac{1}{\mu_{min}}-4\epsilon)}{\ln\mu_{min}} \tag{5.1.1}$$

be a positive constant. Then one can check that for any $s > \chi$, we have

$$\frac{2}{\mu_{min}^{s}} + \frac{4\epsilon\mu_{min}}{\mu_{min}-1} < 1.$$

Thus when $n-k > \chi$, i.e., $k < n-\chi$, we have

$$|h_{k,t}(a)| < \frac{2}{\mu_{min}^{n-k}} + \frac{4\epsilon\mu_{min}}{\mu_{min}-1} < 1.$$

Thus we have

$$h_{k,t}(A_n) \subset D_y(0,1) \subsetneq D_y(0,2)\backslash D_0(t,0).$$

□

**Step 2.** Thus when $a \in A_n$, consider the intersection of $F_{t,a}^{N+\theta n}(D_{\theta n}(t,a))$ with $D_{\vec{x}} \times D_0(t,0)$. Then $\pi_y$ is degree 2 covering map with no branched points onto $D_0(t,0)$. Let $L_{\theta n}(t,a)$ be one of leaves of this covering, then $L_{\theta n}(t,a)$ can be viewed as the graph of a $(m-1)$ vector-valued holomorphic function $l_{\theta n,t,a} : D_0(t,0) \longrightarrow D_{\vec{x}}$, i.e.,

$$L_{\theta n}(t,a) = \{(l_{\theta n,t,a}(z), z) | z \in D_0(t,0)\}.$$

By Lemma 3.0.5, we have

$$\mathrm{dist}(\vec{0}, \pi_{\vec{x}}(L_{\theta n}(t,a))) \leqslant 2C_1|\lambda|^{\theta n}.$$

By the same argument of equation (4.0.3), we have

$$|\pi_y F_{t,a}^{N+n}(l_{\theta n,t,a}(y), y) - \pi_y F_{t,a}^{N+n}((\vec{0}, y))| = O(|\lambda^\theta \mu|^n)$$

where $y \in D_0(t,0)$. Since $|\lambda^\theta \mu| < 1$ and $\widetilde{D}_n(t,a) \subset D_0(t,0)$, for $n$ large enough, we have

$$|\pi_y F_{t,a}^{N+n}(l_{\theta n,t,a}(y), y) - \pi_y F_{t,a}^{N+n}((\vec{0}, y))| < \frac{1}{3}\epsilon$$

where $y \in \widetilde{D}_n(t,a)$. Since

$$(D_y(0, 2+\epsilon), \widetilde{D}_n(t,a), \pi_y F_{t,a}^{N+n})$$

is a quadratic-like map, by proposition 2.2.4, and by the same argument as in the proof of Lemma 4.0.6, there exists a simply connected domain there exists simply connected domains $T_{\theta,n}(t,a) \subset \widetilde{D}_n(t,a)$ and $A_{\theta n,n} \subset \widetilde{A}_n$ such that

$$(D_y(0,2), T_{\theta,n}(t,a), \pi_y F_{t,a}^{N+n}(l_{\theta n,t,a}(y), y))$$

is a quadratic-like map of $y$ variable with critical value $v(\theta n, n, t, a)$, where $a \in A_{\theta n,n}$. Besides, there exists a univalent map

$$h_{\theta n,n,t} : A_{\theta n,n} \longrightarrow D_y(0,2)$$

defined by $h_{\theta n,n,t}(a) = v(\theta n, n, t, a)$.

By Step 1 and the definition of $L_{\theta n}(t,a)$, we know there exists a simply connected domain $\widetilde{S}_{\theta,n}(t,a) \subset D_{\theta n}(t,a)$ such that $\pi_y F_{t,a}^{N+\theta n}$ is a biholomorphic map from $\widetilde{S}_{\theta,n}(t,a)$ to $T_{\theta,n}(t,a)$ and $F_{t,a}^{N+\theta n}(S_{\theta,n}(t,a))$ is a subset of $L_{\theta n}(t,a)$.

Thus we have the quadratic-like map

$$(D_y(0,2), \widetilde{S}_{\theta,n}(t,a), \pi_y F_{t,a}^{N+n}(F_{t,a}^{N+\theta n}))$$

with same critical value $v(\theta n, n, t, a)$ and function $h_{\theta n,n,t}$ defined above.

**Step 3.** For $n > k \geqslant 0$, let $U_k$ be a neighborhood of $\widetilde{D}_k(t,0)$ such that $U_k \subset U_{k-1}$ and $F_{t,a}^{N+k}(U_k) \supsetneq D_y(0, 2+2\epsilon)$, for $a \in \widetilde{A}_n$. We have the estimation

$$\operatorname{diam}(U_k) \asymp \frac{1}{|\mu|^{\frac{k}{2}}}.$$

Denote $B(k,\theta,n,t)$ to be

$$B(k,\theta,n,t) = h_{\theta n,n,t}^{-1}(U_k).$$

We have the following estimations:

**Lemma 5.1.2.** *When $n$ large enough, we have*

$$\operatorname{diam}(B(k,\theta,n,t)) \asymp \frac{1}{|\mu|^{n+k}}. \tag{5.1.2}$$

*Besides, there exists an uniform constant $C > 0$ such that when $k$ is large enough, we have*

$$\frac{1}{C|\mu|^n} < |a| < \frac{C}{|\mu|^n} \tag{5.1.3}$$

*for every $a \in B(k,\theta,n,t)$.*

*Proof.* The first estimation follows from the distortion theorem 2.1.2. Since $h_{\theta n,n,t}(0) = 0$, when $k$ large enough, there exist some $\epsilon \in (0,1)$, such that

$$U_k \subset D(0, 1+\epsilon) \setminus D(0, 1-\epsilon). \tag{5.1.4}$$

Then by distortion theorem 2.1.2, we can see the second estimation. □

When $a \in B(k,\theta,n,t)$, we know that

$$(U_k, (\pi_y F_{t,a}^{N+n}(F_{t,a}^{N+\theta n}))^{-1}(U_k), \pi_y F_{t,a}^{N+n}(F_{t,a}^{N+\theta n}))$$

is a proper map of degree 2, and we denote $(\pi_y F_{t,a}^{N+n}(F_{t,a}^{N+\theta n}))^{-1}(U_k)$ by $S_{k,\theta,n}(t,a)$.

We are interested in the map $\pi_y F_{t,a}^{k+N} \circ F_{t,a}^{N+n} \circ F_{t,a}^{N+\theta n}$ on $S_{k,\theta,n}(t,a)$. First, let us analyze the map $\pi_y F_{t,a}^{k+N} \circ \pi_y F_{t,a}^{N+n} \circ F_{t,a}^{N+\theta n}$ on $S_{\theta,n}(t,a)$.

By above discussion and Lemma 4.0.4, we know that for $a \in B(k,\theta,n,t)$,

$$(\pi_y F_{t,a}^{k+N} \circ \pi_y F_{t,a}^{N+n} \circ F_{t,a}^{N+\theta n}(S_{k,\theta,n}(t,a)), S_{k,\theta,n}(t,a), \pi_y F_{t,a}^{k+N} \circ \pi_y F_{t,a}^{N+n} \circ F_{t,a}^{N+\theta n})$$

is a polynomial-like map of degree 4. It has 3 critical points and critical values (counting with the multiplicities), the critical values are $\pi_y F_{t,a}^{N+k}(v(\theta n, n, t, a))$ and $v(k,t,a)$. As long as $v(\theta n, n, t, a) \neq q_3(k,t,a)$, we have $\pi_y F_{t,a}^{N+k}(v(\theta n, n, t, a)) \neq v(k,t,a)$.

When $a \in \widetilde{A}_n$, choose a simply-connected domain $U_n$, such that the following holds:

$$\widetilde{D}_n(t,a) \subset U_n$$

and

$$\operatorname{diam}(U_n) = O(\frac{1}{|\mu|^{\frac{n}{2}}}).$$

By previous discussion, we know that

$$q_3(k,t,a) \in U_n$$

where $0 \leqslant k \leqslant n$, $a \in \widetilde{A}_n$. Now we prove the following lemma for $\pi_y F_{t,a}^{N+k}$ on $U_k$:

**Lemma 5.1.3.** *There exist a constant $K_0 > 0$ and an integer $\beta > 0$, such that for any $k < n - \beta$, $z \in D_y(0, 2+\epsilon) \backslash D_y(0, \frac{K_0}{|\mu|^{n-k}})$, there exists two univalent functions*

$$c_1(k,z), c_2(k,z) : B(k,\theta,n,t) \longrightarrow U_k,$$

*such that*

$$\pi_y F_{t,a}^{N+k}(\vec{0}, c_i(k,z)(a)) = z,$$

*where $i = 1, 2$. Their images $c_i(k,z)(B(k,\theta,n,t))$ are strictly inside $U_k$. Moreover, for any simply-connected domain $T$ inside $D_y(0, 2+\epsilon) \backslash D_y(0, \frac{K_0}{|\mu|^{n-k}})$, the maps:*

$$z \longrightarrow c_i(k,z)(a)$$

*are univalent on $T$ for any $a \in B(k,\theta,n,t)$.*

*Proof.* Choose a $K_0 > 0$ such that

$$F_{t,a}^{N+k}(U_n) \subset D_y(0, \frac{K_0}{|\mu|^{n-k}})$$

We choose $\beta > 0$, such that

$$D_y(0, \frac{K_0}{|\mu|^{n-k}}) \subset D_y(0, 1)$$

for $k < n - \beta$. Now for any $z \in D_y(0, 2 + \epsilon) \backslash D_y(0, \frac{K_0}{|\mu|^{n-k}})$, consider the equation

$$\pi_y F_{t,a}^{N+k}((\vec{0}, y)) = z,$$

where $a \in B(k, \theta, n, t)$. Then it has 2 different solutions on $y$ variable $y = c_1$ and $y = c_2$. In this situation, we have

$$\left. \frac{\partial \pi_y F_{t,a}^{N+k}}{\partial y} \right|_{y=c_i} \neq 0,$$

where $i = 1, 2$, then by implicit function theorem, we have

$$y = c_i(k, z)(a)$$

are well defined holomorphic functions on the set $B(k, \theta, n, t)$, where $i = 1, 2$.

For any fixed $z \in D_y(0, 2 + \epsilon) \backslash D_y(0, \frac{K_0}{|\mu|^{n-k}})$, we have that $c_i$ is univalent on $a$-variable because

$$\frac{\partial c_i}{\partial a} = -\frac{\partial \pi_y F_{t,a}^{N+k}}{\partial a} / \frac{\partial \pi_y F_{t,a}^{N+k}}{\partial y} \neq 0.$$

When $y \in \partial U_k$, since $F_{t,a}^{N+k}(U_k) \supsetneq D_y(0, 2 + 2\epsilon)$, we know that $F_{t,a}^{N+k}(\partial U_k)$ lies outside of $D_y(0, 2 + 2\epsilon)$. Thus $c_i(k, z)(B(k, \theta, n, t))$ are strictly inside $U_k$.

Let $T$ be a simply-connected domain inside $D_y(0, 2 + \epsilon) \backslash D_y(0, \frac{K_0}{|\mu|^{n-k}})$, for any fixed $a \in B(k, \theta, n, t)$, again we consider the equation

$$\pi_y F_{t,a}^{N+k}((\vec{0}, y)) = z$$

where $z \in T$. For different $z$, we must have different $y$, then the conclusion follows easily. □

Now we only consider $k$ satisfying $0 \leqslant k < n - \beta$.

Given $z \in D_y(0, 2+\epsilon) \backslash D_y(0, \frac{K_0}{|\mu|^{n-k}})$, for $i = 1, 2$, define the maps:

$$\widetilde{c}_i(k, z) : U_k \longrightarrow U_k$$

by

$$\widetilde{c}_i(k, z)(y) = c_i(k, z)(h^{-1}_{\theta n,n,t}(y))$$

By lemma 5.1.3, $\widetilde{c}_i(k, z)(U_k)$ lies strictly inside $U_k$. Thus by theorem 2.3.1, there exist an unique fixed point $y_i(k, z)$ such that

$$\widetilde{c}_i(k, z)(y_i(k, z)) = y_i(k, z).$$

Moreover, $y_i(k, z)$ depends holomorphically on $z$-variable when $z$ lies in any any simply-connected domain inside $D_y(0, 2+\epsilon) \backslash D_y(0, \frac{K_0}{|\mu|^{n-k}})$. Denote $\widetilde{b}_i(k)(z) = h_{\theta n,n,t}(y_i(k, z))$. Thus we have

$$\begin{aligned}
\pi_y F^{N+k}_{t,\widetilde{b}_i(k)(z)}(v(\theta n, n, t, \widetilde{b}_i(k)(z))) &= \pi_y F^{N+k}_{t,\widetilde{b}_i(k)(z)} h_{\theta n,n,t}(\widetilde{b}_i(k)(z)) \\
&= \pi_y F^{N+k}_{t,\widetilde{b}_i(k)(z)}(y_i(k, z)) \\
&= \pi_y F^{N+k}_{t,\widetilde{b}_i(k)(z)}(\widetilde{c}_i(k, z)(y_i(k, z))) \\
&= \pi_y F^{N+k}_{t,\widetilde{b}_i(k)(z)}(c_i(k, z)(\widetilde{b}_i(k)(z)) \\
&= z.
\end{aligned}$$

It is then easy to see that $\widetilde{b}_i(k)(z)$ is an univalent function on $z$ when $z$ lies in any any simply-connected domain inside $D_y(0, 2+\epsilon) \backslash D_y(0, \frac{K_0}{|\mu|^{n-k}})$.

Now we goes back to the map $\pi_y F^{N+k}_{t,a} \circ F^{N+n}_{t,a} \circ F^{N+\theta n}_{t,a}$. Since for any $y \in S_{k,\theta,n}(t, a)$, we have

$$|\pi_{\vec{x}} F^{N+n}_{t,a} \circ F^{N+\theta n}_{t,a}(y)| \leqslant 2C_1 |\lambda|^n.$$

Thus we have

$$|\pi_y F^{N+k}_{t,a} \circ F^{N+n}_{t,a} \circ F^{N+\theta n}_{t,a}(y) - \pi_y F^{N+k}_{t,a} \circ \pi_y F^{N+n}_{t,a} \circ F^{N+\theta n}_{t,a}(y)| \leqslant C_4 |\lambda|^n |\mu|^k. \tag{5.1.5}$$

where $C_4$ is a positive constant. Denote

$$TS_{k,\theta,n}(t,a) = (\pi_y F_{t,a}^{N+k} \circ F_{t,a}^{N+n} \circ F_{t,a}^{N+\theta n})^{-1}(D_y(0, 2+\epsilon))$$

When $n$ large enough such that $100C_4|\lambda|^n|\mu|^{n-\beta} \leqslant \epsilon$, by proposition 2.2.4, we know that

$$(D_y(0, 2+\epsilon), TS_{k,\theta,n}(t,a), \pi_y F_{t,a}^{N+k} \circ F_{t,a}^{N+n} \circ F_{t,a}^{N+\theta n})$$

is a polynomial-like map of degree 4, where $a \in B(k, \theta, n, t)$. We are interested in the critical value of the map $\pi_y F_{t,a}^{N+k} \circ F_{t,a}^{N+n} \circ F_{t,a}^{N+\theta n}$ corresponding to the critical value $\pi_y F_{t,a}^{N+k}((\vec{0}, v(\theta n, n, t, a)))$ of the map $\pi_y F_{t,a}^{N+k} \circ \pi_y F_{t,a}^{N+n} \circ F_{t,a}^{N+\theta n}$, we denote it as $\mathrm{sv}(k, \theta, n, t, a)$.

Denote

$$\widetilde{B}_{k,\theta n,n}(t) = h_{\theta n,n,t}^{-1}(U_k \backslash U_n).$$

Consider the disc $D_y(q_2(t), 10C_4|\lambda|^n|\mu|^k)$, then it is a simply-connected domain inside $D_y(0, 2+\epsilon) \backslash D_y(0, \frac{K_0}{|\mu|^{n-k}})$. We conclude

$$y_i(k, z) \in U_k \backslash U_n$$

for $z \in D_y(q_2(t), 10C_4|\lambda|^n|\mu|^k)$.

Hence $\widetilde{b}_i(k)(z)$ is univalent on $D_y(q_2(t), 10C_4|\lambda|^n|\mu|^k)$ and

$$\widetilde{b}_i(k)(D_y(q_2(t), 10C_4|\lambda|^n|\mu|^k) \subset \widetilde{B}_{k,\theta n,n}(t).$$

For $a \in \widetilde{b}_i(k)(D_y(q_2(t), 10C_4|\lambda|^n|\mu|^k)$, denote

$$P_k(t,a) \subset (\pi_y F_{t,a}^{N+k} \circ \pi_y F_{t,a}^{N+n} \circ F_{t,a}^{N+\theta n})^{-1}(D_y(q_2(t), 10C_4|\lambda|^n|\mu|^k)$$

be the component containing the critical point. Then $P_k(t,a)$ is homeomorphic to a disk. We have following lemma:

**Lemma 5.1.4.** *For $a \in \widetilde{b}_i(k)(D_y(q_2(t), 5C_4|\lambda|^n|\mu|^k)$, let*

$$\widehat{P}_k(t,a) = (\pi_y F_{t,a}^{N+k} \circ F_{t,a}^{N+n} \circ F_{t,a}^{N+\theta n}\big|_{P_k(t,a)})^{-1}(D_y(\widetilde{b}_i^{-1}(k)(a), 3C_4|\lambda|^n|\mu|^k))$$

*then* $\widehat{P}_k(t,a) \subset P_k(t,a)$ *homeomorphic to a disk and* $\pi_y F_{t,a}^{N+k} \circ F_{t,a}^{N+n} \circ F_{t,a}^{N+\theta n}$ *is a proper map of degree* 2 *from* $\widehat{P}_k(t,a)$ *onto* $D_y(\widetilde{b}_i^{-1}(k)(a), 3C_4|\lambda|^n|\mu|^k)$*. Thus* $\mathrm{sv}(k,\theta,n,t,a)$ *is in the image, i.e., we have*

$$|\,\mathrm{sv}(k,\theta,n,t,a) - (\widetilde{b}_i(k))^{-1}(a)| \leqslant 2C_4|\lambda|^n|\mu|^k \tag{5.1.6}$$

*Let* $SB_i(k,t) \subset \widetilde{b}_i(k)(D_y(q_2(t), 5C_4|\lambda|^n|\mu|^k)$ *be the* $a$ *parameters such that* $\mathrm{sv}(k,\theta,n,t,a) \in D_y(q_2(t), 2C_4|\lambda|^n|\mu|^k)$*. Then we know* $SB_i(k,t)$ *is homeomorphic to a disk and* $\mathrm{sv}(k,\theta,n,t,a)$ *is univalent on* $a$*-variable from* $SB_i(k,t)$ *onto* $D_y(q_2(t), 2C_4|\lambda|^n|\mu|^k)$ *for each* $t$*.*

*Proof.* We know that $\pi_y F_{t,a}^{N+k} \circ \pi_y F_{t,a}^{N+n} \circ F_{t,a}^{N+\theta n}$ is a proper map of degree 2 from $P_k(t,a)$ onto $D_y(q_2(t), 10C_4|\lambda|^n|\mu|^k)$, where

$$a \in \widetilde{b}_i(0)(D_y(q_2(t), 10C_4|\lambda|^n|\mu|^k)).$$

When $a \in \widetilde{b}_i(k)(D_y(q_2(t), 5C_4|\lambda|^n|\mu|^k))$, we know that

$$D_y(\widetilde{b}_i^{-1}(k)(a), 3C_4|\lambda|^n|\mu|^k) \subset D_y(q_2(t), 9C_4|\lambda|^n|\mu|^k).$$

By inequality (5.1.5) and proposition 2.2.3, we know that $\widehat{P}_k(t,a)$ homeomorphic to a disk and $\pi_y F_{t,a}^{N+k} \circ F_{t,a}^{N+n} \circ F_{t,a}^{N+\theta n}$ is a proper map of degree 2 from $\widehat{P}_k(t,a)$ onto $D_y(\widetilde{b}_i^{-1}(0)(a), 3C_4|\lambda|^n|\mu|^k)$. Besides, by proposition 2.2.3, we also have

$$|\,\mathrm{sv}(k,\theta,n,t,a) - (\widetilde{b}_i(k))^{-1}(a)| \leqslant 2C_4|\lambda|^n|\mu|^k \tag{5.1.7}$$

Then again by proposition 2.2.3, we know $SB_i(k,t)$ is homeomorphic to a disk and $\mathrm{sv}(k,\theta,n,t,a)$ is univalent on $a$-variable from $SB_i(k,t)$ onto $D_y(q_2(t), 2C_4|\lambda|^n|\mu|^k)$ for each $t$. □

**Definition 5.1.1.** Denote $b_{n,k,i}(t)$ be the point in $SB_i(k,t)$ such that

$$sv(k,\theta,n,t,b_{n,k,i}(t)) = q_2(t),$$

i.e., $F_{t,b_{n,k,i}(t)}^{N+\theta n+N+n+N+k}$ has a secondary tangency. By lemma 5.1.4, for any $k < n - \beta$, we know that $b_{n,k,i}(t)$ is holomorphically depending on $t$ for $t \in \mathbb{D}_t(\tau)$. Especially, by lemma 5.1.2, there exist uniform constant $C > 0$, such that when $n$ large enough, for $k$ large enough, $i = 1, 2$ and $t \in T(\mu_{min}, \mu_{max})$, we have

$$\frac{1}{C|\mu|^n} < |b_{n,k,i}(t)| < \frac{C}{|\mu|^n}. \tag{5.1.8}$$

**Step 4.** Denote $Pr_{\frac{\partial}{\partial y}} DF_{t,a}^N(\frac{\partial}{\partial x_1})\Big|_{(\vec{x},y)}$ be the $\frac{\partial}{\partial y}$ component of $DF_{t,a}^N(\frac{\partial}{\partial x_1})\Big|_{(\vec{x},y)}$ at $(\vec{x}, y)$. Since $Pr_{\frac{\partial}{\partial y}} DF_{t,a}^N(\frac{\partial}{\partial x_1})\Big|_{q_3(t)}$ is non-zero, there exist an neighborhood of $q_3(t)$, denoted by $S$, such that

$$|Pr_{\frac{\partial}{\partial y}} DF_{t,a}^N(\frac{\partial}{\partial x_1})\Big|_{(\vec{x},y)}| > C > 0 \tag{5.1.9}$$

for every point $(\vec{x}, y) \in S$. For $n$ large enough, we have $U_{\theta n} \subset S$.

Now we consider the case when $a \in D_a(a_n(t), \frac{\eta}{|\mu|^{2n}})$, i.e., when $a$ is in the parameter discs which exhibit a periodic sink. Choose an $a$ in this disc, then consider the map $(D_y(0,2), D_n(t,a), \pi_y F_{t,a}^{N+n})$, we know the critical value $v(n,t,a) \in U_n$. Thus we know there exist a positive constant $e > 0$ such that $D_y(v(n,t,a), e) \subsetneq S \subset D_y(0,2)$. Denote $\phi(y)$ be the affine map transforming $D_y(v(n,t,a), e)$ to $\Delta$ with $v(n,t,a)$ to the origin of $\Delta$, i.e., $\phi(y) = \frac{1}{e}(y - v(n,t,a))$.

Denote $\xi(y)$ be the uniformization map from $(\pi_y F_{t,a}^{N+n})^{-1}(D_y(v(n,t,a), e))$ to $\Delta$ such that $\xi(q_3(n,t,a)) = 0$. Then $\phi \circ \pi_y F_{t,a}^{N+n} \circ \xi^{-1}$ is a holomorphic map from $\Delta$ to itself preserving the boundary. 0 is the only zero and it is of degree 2. Thus $\phi \circ \pi_y F_{t,a}^{N+n} \circ \xi^{-1} = uz^2$ where $|u| = 1$.

Denote the $\phi \circ \pi_y F_{t,a}^{N+n}(l_{\theta n,t,a}(-), -) \circ \xi^{-1}$ be $g(z)$, then it is a holomorphic map from $\Delta$ to $\mathbb{C}$.

**Lemma 5.1.5.** *For $n$ large enough , there exist an uniform constant $C > 0$ such that*

$$\frac{1}{C}|\lambda^\theta \mu|^n < |\pi_y F_{t,a}^{N+n}(l_{\theta n,t,a}(y), y) - \pi_y F_{t,a}^{N+n}((\vec{0}, y))| < C|\lambda^\theta \mu|^n.$$

*where $y \in D_y(v(n,t,a), e)$.*

*Proof.* The upper bound part is already proven in the previous discussions. Now we prove for the lower bound. When $n$ large enough, we know that $F_{t,a}^N(D_{\theta n}(t,a))$ are inside $J_{K_0}$. Thus by lemma 3.0.5, we have

$$L_{\theta n}(t,a)) \subset J_{\gamma^{\theta n} K_0}$$

and

$$\mathrm{dist}(\vec{0}, \pi_{\vec{x}}(L_{\theta n}(t,a))) \geqslant 2C_1(K_0)|\lambda|^{\theta n}.$$

Thus when $n$ large enough, by condition (5.1.9), we have

$$F^N_{t,a}(L_{\theta n}(t,a))) \subset J_{K_0}$$

and

$$|\pi_y F^N_{t,a}(l_{\theta n,t,a}(y), y) - \pi_y F^N_{t,a}((\vec{0}, y))| > 2CC_1(K_0)|\lambda|^{\theta n}.$$

Furthermore, there exist an $\iota > 0$, such that

$$F^N_{t,a}(l_{\theta n,t,a}(y), y) \in V^u(F^N_{t,a}((\vec{0}, y)); \iota)$$

Then by lemma 3.0.7, their exist an uniform constant $C > 0$ such that

$$|\pi_y F^{N+n}_{t,a}(l_{\theta n,t,a}(y), y) - \pi_y F^{N+n}_{t,a}((\vec{0}, y))| > \frac{1}{C}|\lambda^\theta \mu|^n.$$

□

Thus we have

$$\frac{1}{Ce}|\lambda^\theta \mu|^n < |g(z) - uz^2| < \frac{C}{e}|\lambda^\theta \mu|^n.$$

Then by part (2) of lemma 2.2.1, for the case $\epsilon = \frac{C}{e}|\lambda^\theta \mu|^n$, $t = \frac{1}{C^2}$. Since $\epsilon$ goes to zero when $n$ getting larger. there exist $\alpha > 0$ such that $t^*(\epsilon, \alpha) > \frac{1}{C^2}$. Thus we have the lower bound:

$$|\phi \circ h_{\theta n,n,t}(a) \circ \xi^{-1}| > \alpha \frac{1}{Ce}|\lambda^\theta \mu|^n.$$

Then we get our desired estimation:

**Lemma 5.1.6.** *For $a \in D_a(a_n(t), \frac{\eta}{|\mu|^{2n}})$,*

$$\alpha \frac{1}{C}|\lambda^\theta \mu|^n < |h_{\theta n,n,t}(a) - h_{n,t}(a)| < 2C|\lambda^\theta \mu|^n.$$

*Furthermore, by condition 2.5.1, lemma 3.0.5 and 3.0.7, we can conclude that for $k < n$,*

$$\frac{1}{C}|\lambda^\theta \mu|^{2n}|\mu|^k < |F^{N+k}_{t,a}(h_{\theta n,n,t}(a)) - F^{N+k}_{t,a}(h_{n,t}(a))| < C|\lambda^\theta \mu|^{2n}|\mu|^k.$$

Since $h_{n,t}(a) \in U_n$, we know that $h_{\theta n,n,t}(a) \in U_0 \backslash U_n$, we have

$$|sv(0,\theta,n,t,a)| < |sv(1,\theta,n,t,a)| < \cdots < |sv(n-1,\theta,n,t,a)| < |sv(n,\theta,n,t,a)|$$

and

$$|sv(0,\theta,n,t,a)| < 2+\epsilon < |sv(n,\theta,n,t,a)|.$$

**Definition 5.1.2.** Denote $n_0 < n$ be the integer such that

$$sv(n_0,\theta,n,t,a) \in D_y(0,2+\epsilon),$$
$$sv(n_0+1,\theta,n,t,a) \notin D_y(0,2+\epsilon).$$

**Lemma 5.1.7.** *For $a \in D_a(a_n(t), \frac{\eta}{|\mu|^{2n}})$, when $n$ large enough, there exist constant $C' > 0$ independent of $n$ such that*

$$\frac{1}{C'}|\lambda^\theta \mu|^{2n}|\mu|^k < |\operatorname{sv}(k,\theta,n,t,a)| < C'|\lambda^\theta\mu|^{2n}|\mu|^k \tag{5.1.10}$$

*for $0 < k < n$. Furthermore, we have estimation on $n_0$:*

$$n_0 = (\frac{-2\log(|\lambda^\theta\mu|)}{\log(|\mu|)})n + O(1).$$

*Proof.* For $k < n$, we have the following estimations:

$$|\operatorname{sv}(k,\theta,n,t,a) - F_{t,a}^{N+k}(h_{\theta n,n,t}(a))| < 2C_4|\lambda|^n|\mu|^k$$

by lemma 5.1.4,

$$\frac{1}{C}|\lambda^\theta\mu|^{2n}|\mu|^k < |F_{t,a}^{N+k}(h_{\theta n,n,t}(a)) - F_{t,a}^{N+k}(h_{n,t}(a))| < C|\lambda^\theta\mu|^{2n}|\mu|^k$$

by lemma 5.1.6, and

$$|F_{t,a}^{N+k}(h_{n,t}(a))| < C|\frac{1}{\mu^{n-k}}|$$

by the choice of $a$. Since we have

$$\frac{|\lambda|^n|\mu|^k}{|\lambda^\theta\mu|^{2n}|\mu|^k} = \left(\frac{|\lambda|^{1-2\theta}}{|\mu|^2}\right)^n$$
$$\frac{|\frac{1}{\mu^{n-k}}|}{|\lambda^\theta\mu|^{2n}|\mu|^k} = \left(\frac{1}{|\lambda^{2\theta}\mu^3|}\right)^n$$

where

$$\frac{|\lambda|^{1-2\theta}}{|\mu|^2}, \frac{1}{|\lambda^{2\theta}\mu^3|} < 1.$$

Thus when $n$ large enough, there exist a uniform constant $C' > 0$ such that the following two inequality holds:

$$\begin{aligned}
|\operatorname{sv}(k,\theta,n,t,a)| &\leqslant |\operatorname{sv}(k,\theta,n,t,a) - F_{t,a}^{N+k}(h_{\theta n,n,t}(a))| + |F_{t,a}^{N+k}(h_{\theta n,n,t}(a)) - F_{t,a}^{N+k}(h_{n,t}(a))| \\
&\quad + |F_{t,a}^{N+k}(h_{n,t}(a))| \\
&< 2C_4|\lambda|^n|\mu|^k + C|\lambda^\theta\mu|^{2n}|\mu|^k + C|\frac{1}{\mu^{n-k}}| \\
&< C'|\lambda^\theta\mu|^{2n}|\mu|^k,
\end{aligned}$$

$$\begin{aligned}
|\operatorname{sv}(k,\theta,n,t,a)| &\geqslant |F_{t,a}^{N+k}(h_{\theta n,n,t}(a)) - F_{t,a}^{N+k}(h_{n,t}(a))| - |\operatorname{sv}(k,\theta,n,t,a) - F_{t,a}^{N+k}(h_{\theta n,n,t}(a))| \\
&\quad - |F_{t,a}^{N+k}(h_{n,t}(a))| \\
&> \frac{1}{C}|\lambda^\theta\mu|^{2n}|\mu|^k - 2C_4|\lambda|^n|\mu|^k - C|\frac{1}{\mu^{n-k}}| \\
&> \frac{1}{C'}|\lambda^\theta\mu|^{2n}|\mu|^k.
\end{aligned}$$

Thus by the definition of $n_0$, we have

$$\begin{aligned}
\frac{1}{C'}|\lambda^\theta\mu|^{2n}|\mu|^{n_0} &< |sv(n_0,\theta,n,t,a)| < 2+\epsilon \\
&< |sv(n_0+1,\theta,n,t,a)| < C'|\lambda^\theta\mu|^{2n}|\mu|^{n_0+1}.
\end{aligned}$$

Then we have

$$\frac{2+\epsilon}{C'|\mu|} < |\lambda^\theta\mu|^{2n}|\mu|^{n_0} < (2+\epsilon)C'.$$

After taking logarithm and take $n$ large enough, we have

$$n_0 = (\frac{-2\log(|\lambda^\theta\mu|)}{\log(|\mu|)})n + O(1).$$

□

## 5.2 Finite time Collect-Eckmann condition

In this section we will prove that for $a$ in a neighborhood of $b_{n,k,i}(t)$, there are points in the phase space satisfying the finite time Collect-Eckmann condition.

Let us consider $a$ in a neighborhood of $b_{n,k,i}(t)$, now $\mathrm{sv}(k,\theta,n,t,a)$ is the critical value of the map $\pi_y F_{t,a}^{N+\theta n+N+n+N+k}$ we denote the critical point constructed as $q_1'$, then denote the following forward orbits of $q_1'$:

$$\begin{aligned} z_1 &= F^N\left(q_1'\right) \\ z_2 &= F^{\theta n}\left(z_1\right) \\ z_3 &= F^N\left(z_2\right) \\ z_4 &= F^n\left(z_3\right) \\ z_5 &= F^N\left(z_4\right) \\ z_6 &= F^k\left(z_5\right) \end{aligned} \tag{5.2.1}$$

For any $0 \leqslant l \leqslant k$, we compute the differential matrix of $F^{n+N+l}$ at $z_3$:

$$DF^{n+N+l} = \begin{pmatrix} a_{11}(l)\lambda^l\mu^l & a_{12}(l)\lambda^l\mu^l \\ a_{21}(l)\frac{1}{\mu^{k-l}} & a_{22}(l)\mu^l \end{pmatrix} \begin{pmatrix} A & B \\ C & D\Delta x_{z_4} + 2Q\Delta y_{z_4} \end{pmatrix} \begin{pmatrix} a_{11}(n)\lambda^n\mu^n & a_{12}(n)\lambda^n\mu^n \\ a_{21}(n) & a_{22}(n)\mu^n \end{pmatrix} \tag{5.2.2}$$

Thus we have

$$\begin{aligned} DF^n\left(z_3\right)\begin{pmatrix} v_1 \\ 1 \end{pmatrix} &= \begin{pmatrix} a_{11}(n)\lambda^n\mu^n & a_{12}(n)\lambda^n\mu^n \\ a_{21}(n) & a_{22}(n)\mu^n \end{pmatrix}\begin{pmatrix} v_1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \left(a_{11}(n)v_1 + a_{22}(n)\right)\lambda^n\mu^n \\ a_{21}(n)v_1 + a_{22}(n)\mu^n \end{pmatrix} \end{aligned} \tag{5.2.3}$$

and let us denote it as $\begin{pmatrix} L_x\left(v_1\right) \\ L_y\left(v_1\right) \end{pmatrix}$. Then the two components have the following estimations when $|v_1| < 2(\phi_1+\phi_2+1)$ and $n$ large enough:

$$\left|L_x\left(v_1\right)\right| = O(|\lambda\mu|^n) \tag{5.2.4}$$

and there exists $1 < L_1 < L_2$ such that

$$L_1|\mu|^n < |L_y(v_1)| < L_2|\mu|^n. \tag{5.2.5}$$

Since we have $\Delta x_{z_4} = O(|\lambda|^n)$ and there exists positive constants $L_3, L_4$ such that

$$\frac{L_3}{|\mu|^{\frac{n}{2}}} < |\Delta y_{z_4}| < \frac{L_4}{|\mu|^{\frac{k}{2}}} \tag{5.2.6}$$

Then by computation, we have the following

$$DF^{n+N}(z_3)\begin{pmatrix} v_1 \\ 1 \end{pmatrix} = \begin{pmatrix} AL_x(v_1) + BL_y(v_1) \\ CL_x(v_1) + (D\Delta x_{z_4} + 2Q\Delta y_{z_4})L_y(v_1) \end{pmatrix}. \tag{5.2.7}$$

We can see that

$$DF^{n+N}(z_3)\begin{pmatrix} v_1 \\ 1 \end{pmatrix}_x = O(|\mu|^n), \tag{5.2.8}$$

and

$$DF^{n+N}(z_3)\begin{pmatrix} v_1 \\ 1 \end{pmatrix}_y \asymp \Delta y_{z_4}|\mu|^n. \tag{5.2.9}$$

If we go forward in $l$ steps, we have

$$\begin{aligned} DF^{n+N+l}(z_3)\begin{pmatrix} v_1 \\ 1 \end{pmatrix}_x =& [a_{11}(l)(AL_x(v_1) + BL_y(v_1)) \\ &+ a_{12}(l)(CL_x(v_1) + (D\Delta x_{z_4} + 2Q\Delta y_{z_4})L_y(v_1))](|\lambda\mu|)^l \end{aligned}$$

and

$$\begin{aligned} DF^{n+N+l}(z_3)\begin{pmatrix} v_1 \\ 1 \end{pmatrix}_y =& a_{21}(l)\frac{1}{\mu^{k-l}}(AL_x(v_1) + BL_y(v_1)) \\ &+ a_{22}(l)\mu^l(CL_x(v_1) + (D\Delta x_{z_4} + 2Q\Delta y_{z_4})L_y(v_1). \end{aligned}$$

Thus we have

$$|DF^{n+N+l}(z_3)\begin{pmatrix} v_1 \\ 1 \end{pmatrix}_x| = O((|\lambda\mu|)^l|\mu|^n) \tag{5.2.10}$$

and

$$|DF^{n+N+l}(z_3)\begin{pmatrix} v_1 \\ 1 \end{pmatrix}_y| \geq 2|Q||\Delta y_{z_4}||a_{22}(k)|L_1|\mu|^{n+l} - L|\mu|^{n+l-k}. \tag{5.2.11}$$

where $L$ a positive constant. Thus we have the following lemma:

**Lemma 5.2.1.** *There exists constant $C > 0$ such that, when $|\Delta y_{z_4}| > C\frac{1}{|\mu|^{\min\{n/2,k\}}}$, for any $m-1$ dimensional vector $v_1$ with $|v_1| < 2(\phi_1 + \phi_2 + 1)$ and $0 \leqslant l \leqslant k$, we have*

$$|DF^{n+N+l}(z_3)\begin{pmatrix} v_1 \\ 1 \end{pmatrix}_y| > (L+1)|\mu|^{n+l-k}. \tag{5.2.12}$$

*More concretely ,we have*

$$|DF^{n+N+l}(z_3)\begin{pmatrix} v_1 \\ 1 \end{pmatrix}_y| \asymp |\Delta y_{z_4}||\mu|^{n+l}. \tag{5.2.13}$$

*Besides there exist some constant $C' > 0$ such that $DF^{n+N+k}(z_3)\begin{pmatrix} v_1 \\ 1 \end{pmatrix}$ are in the cone $\{(\vec{v}_x, v_y)||\vec{v}_x| < C'(|\lambda\mu|)^k|v_y|\}$, and when $k$ large enough, we have $C'(|\lambda\mu|)^k < \phi_1$.*

Since by the construction of $z_4$, we know $z_4 \in U_k \backslash U_n$, and then we can find a simply-connected domain $U_c$ between $U_k$ and $U_n$ such that any point $z \in U_k \backslash U_c$, we have $|\Delta z| > C\frac{1}{|\mu|^{\min\{n/2,k\}}}$, and we denote

$$CE(n,k) := h^{-1}_{\theta n,n,t}(U_k \backslash U_c).$$

It is an annulus with same modulus with $U_k \backslash U_c$, and when $a \in CE(n,k)$, inequality 5.2.12 holds. And this would be the neighborhood of $b_{n,k,i}(t)$ we now taking. By previous constructions, we can find a number $\beta \in (0,1)$ such that

$$\frac{k}{n} < \beta \tag{5.2.14}$$

for all admissible $(n,k)$ pairs when $n$ large enough. Then we let

$$\rho = |\mu_{min}|^{\min\{\frac{2}{3}(1-\beta),\alpha\}}, \tag{5.2.15}$$

where $\alpha \in (0,1)$ is a constant satisfies

$$\alpha < \frac{1 - \frac{M}{n}\frac{\log \frac{1}{s}}{\log |\mu|_{min}}}{1 + \frac{k}{n} + \frac{N+M}{n}} \tag{5.2.16}$$

for any

$$n > 2N + 2\left(1 + \frac{\log \frac{1}{s}}{\log |\mu|_{min}}\right) M. \tag{5.2.17}$$

Thus when (5.2.17) holds, we have

$$\rho \leqslant \min_{n \geq 2N, k, 0 \leqslant l \leqslant k, \mu} |\mu|^{\frac{n+l-k}{n+N+l}} \tag{5.2.18}$$

and

$$|\mu|^n s^l \geqslant \rho^{n+N+k+l} \tag{5.2.19}$$

for any $0 \leqslant l \leqslant M$. Now we can state our key lemma:

**Lemma 5.2.2.** *Let any $a \in CE(n,k)$, integer $T \in (0, n+N+k]$, there exist constant $C > 1$, such that for any $m-1$ dimensional vector $v_1$ with $|v_1| < 2(\phi_1 + \phi_2 + 1)$, we have*

$$||DF^T(z_3)\begin{pmatrix} v_1 \\ 1 \end{pmatrix}|| \geq |DF^T(z_3)\begin{pmatrix} v_1 \\ 1 \end{pmatrix}_y| > C\rho^T. \tag{5.2.20}$$

*when $n$ large enough. Furthermore, when $z_6$ are in the neighborhood of $q_2$ as in the condition (1) states, then we can extend (5.2.20) further to the case $T \in [n+N+k, n+N+k+M]$.*

*Proof.* By inequality 5.2.12 and the fact 5.2.18, we know condition (5.2.20) is true for $T \in [n+N, n+N+k]$. And for $T \in (0,n]$, by lemma 3.0.8, we have

$$||DF^T(z_3)\begin{pmatrix} v_1 \\ 1 \end{pmatrix}|| \geq |DF^T(z_3)\begin{pmatrix} v_1 \\ 1 \end{pmatrix}_y| > C_1|\mu|^T > C_1'\rho^T \tag{5.2.21}$$

with $C_1 > 0$ and $C_1' > 1$ when $n$ large enough. The last part of $T$ is when $T \in [n+1, n+N-1]$, when $n$ large enough such that

$$\frac{\rho^{n+N}}{|\mu|^n} < s^N \frac{C_1'}{C_1} \tag{5.2.22}$$

We have that

$$||DF^T(z_3)\begin{pmatrix} v_1 \\ 1 \end{pmatrix}|| > C_1|\mu|^n s^{T-n} > C_1'\rho^{N+n} \geq C_1'\rho^T. \tag{5.2.23}$$

Taking $C$ to be the smallest constant appeared in all above estimation, we still have $C > 1$. now let $z_6$ are in the neighborhood of $q_2$ as in the condition (1) states, then for $T \in [n+N+k, n+N+k+M]$, we have

$$||DF^T(z_3)\begin{pmatrix} v_1 \\ 1 \end{pmatrix}|| > C|\mu|^n s^{T-n-N-k} > C\rho^T \tag{5.2.24}$$

where $C > 1$ and the last inequality hold by (5.2.19). □

# Chapter 6

# The renormalization scheme and the Collet-Eckmann condition

Since now we have constructed a secondary tangency, it is natural to localize ourselves to this new homoclinic tangency and its unfolding, we will describe this renormalization procedure first, then we will prove that the limiting object, which is a codimension 1 Lamination, will satifies the Collet-Eckmann condition.

## 6.1 The renormalization scheme

**Definition 6.1.1.** Let $U$ be a open connected subset of $\mathbb{C}^n$, for a holomorphic mapping $f : U \longrightarrow \mathbb{C}$, let $U_1$ an open connected subset in $U$, $\delta > 0$, define the $\delta$-cylindrical neighborhood of $f$ over $U_1$ in $\mathbb{C}^{n+1}$, denoted by $CY(f, U_1)$, as

$$CY(f, U_1; \delta) := \bigcup_{z \in U_1} \{z\} \times D(f(z), \delta) \tag{6.1.1}$$

where $D(f(z), \delta)$ is the disc in $\mathbb{C}$ with center $f(z)$ and radius $\delta$. We endow it with natural subspace topology.

It has the following properties:

**Proposition 6.1.1.** *$CY(f, U_1; \delta)$ has a complex 1-dimensional foliation with leaves $D(f(z), \delta)$ for $z \in U_1$, it also carries a complex n-dimensional foliation with leaves $Gr(f|_{U_1} + w)$ with $w \in D(0, \delta)$.*

**Definition 6.1.2.** Let $b : T_\mu(\mu_{min}, \mu_{max}) \times T_\lambda(\lambda_{min}, \lambda_{max}) \times U \longrightarrow \mathbb{C}$ be a holomorphic mapping, $\delta > 0$, we say a family $F : P \times M \longrightarrow M$ is an unfolding of strong homoclinic tangency with respect to $(P, b)$ the following holds:

(1). There exists some $\delta > 0$, such that $P = CY(b; \delta)$;

(2) Let $G$ be the following family

$$G : T_\mu(\mu_{min}, \mu_{max}) \times T_\lambda(\lambda_{min}, \lambda_{max}) \times U \times D(0, \delta) \times M \longrightarrow M$$

$$(t, a, x) \longrightarrow F_{t,a+b(t)}(x).$$

This family $G$ is an unfolding of strong homoclinic tangency.

Now using above definition, we can summarize the secondary tangency as the following lemma:

**Lemma 6.1.2.** *Let $b_{n,k,i}$ be the mapping of seconday tangency, then there exist a small cylinder neighborhood of the graph of $b_{n,k,i}$, denoted as $P = CY(B : \delta)$, such that $F^{N+\theta n+N+n+N+k+M}$ is an unfolding of strong homoclinic tangency with respect to $(P, b_{n,k,i})$. Furthermore, the new tangency is again in the neighborhood of $q_1$, which means we can keep using the same normalization in the phase space.*

Thus if we start with some unfolding of strong homoclinic tangency $(P^{(1)}, b^{(1)})$ by keep using lemma 6.1.2, and in each level, we choose a new $b_{n,k,i}$ map to create new secondary tangency, we can have a chain of renormalizations $(P^{(l)}, b^{(l)})$ with $P^{(l+1)} \subset P^{(l)}$. After all, let us consider the intersection of all $P^{(l)}$, this limiting object is a graph of some holomorphic map over $T_\mu(\mu_{min}, \mu_{max}) \times T_\lambda(\lambda_{min}, \lambda_{max}) \times U$ in the parameter space, we denote this map as $b_\infty$.

Now if we collect all possible chains of renormalizations, we end up with a collection of limiting objects $b_\infty$, by the construction, we can see that all such $b_\infty$ maps have disjoint graphs in the parameter space. Thus it forms a complex codimention 1 lamination, let us denote it as $\mathfrak{CE}$.

**Proposition 6.1.3.** *$\mathfrak{CE}$ is a codimention-1 lamination in the parameter space $(\lambda, \mu, a) \in T_\lambda \times T_\mu \times D_a$. Every leaf of the lamination is a graph over $T_\lambda \times T_\mu$.*

## 6.2 The Collet-Eckmann condition

In this section we will prove that all the leaves in $\mathfrak{CE}$ satisfy the Collet-Eckmann condition for the whole time in the phase space.

Now suppose we start from an unfolding $(P, b)$. We rename the notation in section 5.2 as the following: Denote $q_1^{(1)}$ to be $q_1'$, $c^{(1)}$ to be $z_3$, $T^{(1)} = n + N + k + M$, $N^{(1)} = N + \theta n + N + n + N + k + M = N + \theta n + N + T^{(1)}$ and $q_3^{(1)}$ to be $F^{N^{(1)}}(q_1^{(1)})$, then we can choose $\delta > 0$ such that $CY(b, \delta) \subset \bigcup_t CE(n, k)$. And now $F$ will of strong homoclinic tangency with respect to $(P', b)$, where $P' = CY(b; \delta)$ (shrink $\delta$ if necessary). Then we can repeat out construction with respect to this new unfolding.

Thus we can find new $n^{(2)}, k^{(2)}$ and corresponding secondary tangency curve $b^{(1)}_{n^{(2)},k^{(2)},i}(t)$, and new $CE^{(1)}(n^{(2)}, k^{(2)})$ such that lemma 5.2.2 holds again for $a \in CE^{(1)}(n^{(2)}, k^{(2)})$, $l \in (0, n^{(2)} + N^{(1)} + k^{(2)})$, and $z_3^{(1)} = F^{N^{(1)} + \theta n^{(2)} + N^{(1)}}(q_1^{(1)\prime})$. Since $N^{(1)} = N + \theta n + N + n + N + k + M$, thus the neighborhood of $c^{(1)}$ will map diffeomorphically onto a neighborhood of $z_3^{(1)}$ through $F^{n+N+k+M}$ when $a \in CE^{(1)}(n^{(2)}, k^{(2)})$. Now let $c^{(2)} = F^{-(n+N+k+M)}(z_3^{(1)})$. Since when $n^{(2)}$ goes to $\infty$, $c^{(2)}$ and $z_3^{(1)}$ will converge to $c^{(1)}$ and $q_3^{(1)}$, we may take $n^{(1)})$ large enough such that $DF^{n+N+k+M}\left(c^{(2)}\right)$ will map the tangent cone $\{(v_1, v_2)||v_1| < 2(\phi_1 + \phi_2 + 1)|v_2|\}$ at $c^{(2)}$ into the tangent cone $\{(v_1, v_2)||v_1| < 2(\phi_1 + \phi_2 + 1)|v_2|\}$ at $z_3^{(1)}$. Thus $c^{(2)}$ will also satisfies lemma 5.2.2 for $0 \leqslant T \leqslant n + N + k + M + n^{(2)} + N^{(1)} + k^{(2)} + M$. We have $T^{(2)} = n + N + k + M + n^{(2)} + N^{(1)} + k^{(2)} = T^{(1)} + n^{(2)} + N^{(1)} + k^{(2)} + M$. Now we can present the induction steps in the following lemma:

**Lemma 6.2.1.** *Suppose we have an unfolding with respect to* $(P^{(l)}, b^{(l)})$*, where where* $P^{(l)} = CY(b^{(l)}; \delta^{(l)})$*, and* $q_{1,3}^{(l)}$ *in the neighborhood of* $q_{1,3}$ *respectively with* $F^{N^{(l)}}(q_1^{(l)}) = q_3^{(l)}$*,* $c^{(l)}$ *in the nested neighborhoods of all previous* $c^{(k)}$*, where* $k < l$*. And* $T^{(l)}$ *such that* $DF^{T^{(l)}}(c^{(l)})$ *satisfies lemma* 5.2.2 *with* $0 \leqslant T \leqslant T^{(l)}$ *when* $a \in P^{(l)}$*.*

*Then for* $(n^{(l+1)}, k^{(l+1)})$ *large enough, we have* $b_{n^{(l+1)},k^{(l+1)},i}^{(l)}(t)$ *curves which is inside* $CE^{(l)}(n^{(l+1)}, k^{(l+1)}) \subset P^{(l)}$*, such that* $z_3^{(l)}$ *satisfies (5.2.2) for* $0 \leqslant T \leqslant n^{(l+1)} + N^{(l)} + k^{(l+1)} + M$*. Then we let*

$$N^{(l+1)} = N^{(l)} + \theta n^{(l+1)} + N^{(l)} + n^{(l+1)} + N^{(l)} + k^{(l+1)} + M = N^{(l)} + \theta n^{(l+1)} + N^{(l)} + T^{(l)} \quad (6.2.1)$$

*and we have* $q_{1,3}^{(l+1)}$ *in the neighborhood of* $q_{1,3}$ *respectively with* $F^{N^{(l+1)}}(q_1^{(l+1)}) = q_3^{(l+1)}$*.* $c^{(l+1)} = F^{-(T^{(l)})}(z_3^{(l)})$ *in the nested neighborhoods of all previous* $c^k$*, where* $k < l + 1$*. Let*

$$T^{(l+1)} = T^{(l)} + n^{(l+1)} + N^{(l)} + k^{(l+1)} + M, \quad (6.2.2)$$

*then* $c^{(l+1)}$ *will also satisfies lemma 5.2.2 for* $0 \leqslant T \leqslant T^{(l+1)}$*.*

Now choose a leaf $b_\infty$ in the lamination $\mathfrak{CE}$, let the graph of $b_\infty(t)$ be $\cap_l P^{(l)}$, then we have the following proposition:

**Lemma 6.2.2.** *For every parameter* $(t, a)$ *in the graph* $b_\infty$*, there exist a point* $c_\infty(t, a)$ *in the intersection of neighborhood of* $c^{(l)}$ *such that it satisfies 5.2.2 for every positive integer* $T$*. Moreover,* $c_\infty(t, a)$ *is a quasi-critical point as definition 2.5.1.*

*Proof.* Now let $a = b_\infty(t)$, by previous constructions, we have $c^{(l)}(t, a)$ and their neighborhoods $U^{(l)}$ satisfying lemma 5.2.2 for $0 \leqslant T \leqslant T^{(l)}$. So we have a nested sequence of closed sets $\{U^{(l)}\}$, then we define $c_\infty(t, a)$ as the intersection of all $U^{(l)}$, i.e.,

$$c_\infty(t, a) = \cap_l U^{(l)}. \quad (6.2.3)$$

It is easy to see $c_\infty(t, a)$ satisfies the lemma 5.2.2 for whole positive time, it is easy to see this point is a quasi-critical point by above construction. Thus we finish the proof. □

*Remark* 6.2.3. By the normalization of unfoldings, there exists neighborhoods $U_1, U_2, U_3$ of $q_1, q_2, q_3$ respectively such that for the whole family of maps, we have

$$F^N(U_3) \subset U_1, F^M(U_2) \subset U_1. \tag{6.2.4}$$

Then the forwards orbit of $c_\infty(t, a)$ under $F_{t,a}$ can be decomposed by the following four types of orbits:

Type $A_n$: An orbit $z, F(z), \ldots, F^n(z)$ with $z \in U_1$, $F^n(z) \in U_3$, $F^i(z) \in D$, for $0 \leqslant i \leqslant n$.

Type $B_k$: An orbit $z, F(z), \ldots, F^k(z)$ with $z \in U_1$, $F^n(z) \in U_2$, $F^i(z) \in D$, for $0 \leqslant i \leqslant k$.

Type $C$: An orbit $z, F(z), \ldots, F^N(z)$ with $z \in U_3$, $F^N(z) \in U_1$.

Type $D$: An orbit $z, F(z), \ldots, F^M(z)$ with $z \in U_2$, $F^N(z) \in U_1$.

IF we may construct a kneading sequence of $F^i_{b_\infty}(c(t, a))$ following lemma 6.2.1, using symbols $A_n, B_k, C, D$ if the part of orbit belongs to corresponding Type above. By the induction formulas of $T^{(l)}$ and $N^{(l)}$, this kneading sequence will be the same when $c(t, a)$ moves along the leaf of the lamination $\mathfrak{CE}$.

## 6.3 Proof of theorem A

Now we may give the proof of theorem A:

*Proof.* Part (1), (2) follow from proposition 6.1.3, Part (3) follows from lemma 6.2.2 and remark 6.2.3. For part (4), notice that for any two leaves $L_1$ and $L_2$ of $\mathfrak{CE}$, by the construction of the lamination, there exist an integer $l \geqslant 0$, such that the $(n^l, k_l)$ of the two leaves differ from each other, which implies that the corresponding $P^{(l)}$ for $L_1$ and $L_2$ have disjoint neighborhoods. Thus if we fixed some $t$, the vertical transversal slice of $\mathfrak{CE}$ over $t$ is totally disjoint. The perfectness follows from the following fact in the renormalization scheme:

For the unfolding of homoclinic tangency with respect to $(P, b)$, we have the

$$\lim_{n+k \longrightarrow \infty} b_{n,k,i}(t) = b(t) \tag{6.3.1}$$

uniformly. Thus for any cylindrical neighborhood of the a leaf $L$ in $\mathfrak{CE}$, there exist another leaf inside this neighborhood. Thus any point in the transversal vertical slice of the $\mathfrak{CE}$ is non-isolated point, which prove the perfectness of the transversal vertical slice of the $\mathfrak{CE}$. This finish the proof of part (4). Part (5) actually comes from the construction of Collet-Eckmann point. Let $t$ be a parameter in a leaf $L$ of $\mathfrak{CE}$, by lemma 6.2.1, we have

$$\lim_{l \longrightarrow \infty} c(t)^{(l)} = c(t). \tag{6.3.2}$$

We also know from construction, that the forward image of $c(t)^{(l)}$ will pass the neighborhood of $c(t)^{(l+1)}$ for every $l > 0$. Thus we can see that $c(t) \in \omega(c(t))$, i.e., $c(t)$ is a recurrent point. □

# Chapter 7

# Coexistence of sink and secondary tangency

In this section, we will consider the coexistence of sink and secondary tangency, by previous discussion, we will restrict ourselves into the case of the intersection of graphs of two maps $a_n(t)$ and $b_{n,k,i}(t)$. This intersection can be furthermore transformed into the solution of the equation

$$sv(n_0, \theta, n, t, a_n(t)) = q_2. \tag{7.0.1}$$

We will discuss above equation in the rest of this section.

The following theorem of Huber[Hub51] is very useful, for a proof we recommend [Kob70](P14. theorem 6.1):

**Theorem 7.0.1.** *Let $T_i$ be the annuli $\{z|0 < r_i < |z| < R_i\}$, i=1,2. Then a holomorphic map $f$ from $T_1$ into $T_2$ satisfies:*

$$|deg(f)| \leqslant \frac{\log \frac{R_2}{r_2}}{\log \frac{R_1}{r_1}}. \tag{7.0.2}$$

The degree in the theorem is the degree of the homomorphism $f_* : \pi_1(T_1) \longrightarrow \pi_1(T_2)$ induced from the map $f$, i.e, if we denote the generators of $\pi_1(T_i)$ by $\alpha_i$, $i = 1, 2$, then we have

$$f_*\alpha_1 = deg(f)\alpha_2.$$

First of all, we consider the case when we fix a $\lambda$ and $\tau$, we prove the following proposition:

**Proposition 7.0.2.** *For every $(\lambda, \tau) \in T_\lambda \times U$, when $n$ large enough, there exist an integer $d_n \in [-\frac{2\log C}{\log \frac{\mu_{max}}{\mu_{min}}}, \frac{2\log C}{\log \frac{\mu_{max}}{\mu_{min}}}]$ such that the following holds:*

*(1) If $k \in [k_0(n), k_1(n)]$, then we know that $\mathrm{sv}(k, \theta, n, t, a_n(t))$ defines a covering map from a subset of $T(\mu_{min}, \mu_{max})$ which is a topological annuli onto the annuli $T(C\mu_{min}^{2n+k}|\lambda|^{2\theta n}, \frac{1}{C}\mu_{max}^{2n+k}|\lambda|^{2\theta n})$ containing $q_2$ with degree $2n + k + d_n$.*

*(2) If $k \in [k_1(n), k_2(n)]$, then we know that $\mathrm{sv}(k, \theta, n, t, a_n(t))$ defines a covering map from a subset of $T(\mu_{min}, t_{n,k})$ which is a topological annuli onto the annuli $T(C\mu_{min}^{2n+k}|\lambda|^{2\theta n}, \frac{L}{C^2})$ containing $q_2$ with degree $2n + k + d_n$.*

*Thus when $k \in [k_0(n), k_2(n)]$, the equation*

$$\mathrm{sv}(k, \theta, n, t, a_n(t)) = q_2 \tag{7.0.3}$$

*has $2n + k + d_n$ solutions.*

*We also have*

$$\lim_n \frac{k_0(n)}{n} = \lim_n \frac{k_1(n)}{n} = 2\theta \frac{\log \frac{1}{|\lambda|}}{\log \mu_{max}} - 2, \tag{7.0.4}$$

*and*

$$\lim_n \frac{k_2(n)}{n} = 2\theta \frac{\log \frac{1}{|\lambda|}}{\log \mu_{min}} - 2. \tag{7.0.5}$$

*Proof.* Whenever $\mathrm{sv}(k+1, \theta, n, t, a_n(t))$ is inside $D_y$, by condition (3.0.3), (3.0.4) and proposition (2.2.3), we have

$$|\,\mathrm{sv}(k+1, \theta, n, t, a_n(t)) - \mu \cdot \mathrm{sv}(k, \theta, n, t, a_n(t))| < 2\min\{M|\lambda|^k, \epsilon\}|\,\mathrm{sv}(k, \theta, n, t, a_n(t))| \tag{7.0.6}$$

Besides, by lemma 5.1.7, when $\mathrm{sv}(k, \theta, n, t, a_n(t))$ is inside $D_y$, it can be written as

$$\mathrm{sv}(k, \theta, n, t, a_n(t)) = K_{n,k}(t)\left(\lambda^\theta \mu\right)^{2n} \mu^k, \tag{7.0.7}$$

where $K_{n,k}(t)$ is holomorphic with uniform bound:

$$\frac{1}{C} < |K_{n,k}(t)| < C.$$

Put formula (7.0.7) into (7.0.6), we have

$$|K_{n,k+1}(t) - K_{n,k}(t)| < \frac{2}{|\mu|}\min\{M|\lambda|^k, \epsilon\}|K_{n,k}(t)|. \tag{7.0.8}$$

First we consider the case $k \leqslant k_1(n)$, where $k_1(n)$ is the largest integer such that the following holds:

$$C\mu_{max}^{2n+k_1(n)}|\lambda|^{2\theta n} < L, \tag{7.0.9}$$

i.e., we have

$$k_1(n) = \lfloor \frac{\log\frac{L}{C} + 2\theta n \log\frac{1}{|\lambda|}}{\log \mu_{max}} - 2n \rfloor \tag{7.0.10}$$

Then by equation (7.0.7) and the condition on $K_{n,k}(t)$, we know that $\mathrm{sv}(k, \theta, n, t, a_n(t))$ is a holomorphic map from $T(\mu_{min}, \mu_{max})$ into annuli $T(\frac{1}{C}\mu_{min}^{2n+k}|\lambda|^{2\theta n}, C\mu_{max}^{2n+k}|\lambda|^{2\theta n})$ inside $D_y$. By Theorem 7.0.1, we know that $|deg(K_{n,k}(t))| \leqslant \frac{2\log C}{\log\frac{\mu_{max}}{\mu_{min}}}$. Since two holomorphic maps from annuli $A_1$ into annuli $A_2$ with same degree are homotopic. Then we may choose a homotopy from $K_{n,k}(t)$ to $C_k^{'}t^{degK_{n,k}(t)}$ where $C_k^{'}$ is a scaling constant. Then we have a homotopy from $\mathrm{sv}(k, \theta, n, t, a_n(t))$ to $C_k^{'}t^{degK_{n,k}(t)+2n+k}\lambda^{2\theta n}$, thus the degree of $\mathrm{sv}(k, \theta, n, t, a_n(t))$ is $2n + k + degK_{n,k}(t)$, which is an integer inside $[2n + k - \frac{2\log C}{\log\frac{\mu_{max}}{\mu_{min}}}, 2n + k + \frac{2\log C}{\log\frac{\mu_{max}}{\mu_{min}}}]$.

We know that for any $t_0 \in [\mu_{min}, \mu_{max}]$, the image of the circle $S(t_0) = \{t||t| = t_0\}$ under $\mathrm{sv}(k, \theta, n, t, a_n(t))$ is a loop inside an annuli $T(\frac{1}{C}t_0^{2n+k}|\lambda|^{2\theta n}, Ct_0^{2n+k}|\lambda|^{2\theta n})$.

Thus when $n$ large enough, the image of $T(\mu_{min}, \mu_{max})$ under $\mathrm{sv}(k, \theta, n, t, a_n(t))$ contains an annuli $T(C\mu_{min}^{2n+k}|\lambda|^{2\theta n}, \frac{1}{C}\mu_{max}^{2n+k}|\lambda|^{2\theta n})$.

Thus $q_2 \in T(C\mu_{min}^{2n+k}|\lambda|^{2\theta n}, \frac{1}{C}\mu_{max}^{2n+k}|\lambda|^{2\theta n})$ is equivalent to

$$C\mu_{min}^{2n+k}|\lambda|^{2\theta n} < 2 < \frac{1}{C}\mu_{max}^{2n+k}|\lambda|^{2\theta n}$$

The first part of the inequality is always satisfied when $k \leqslant k_1(n)$, and the second part of the inequality is equivalent to $k \leqslant k_0(n)$, where $k_0(n)$ is the smallest number such that the inequality $2 < \frac{1}{C}\mu_{max}^{2n+k}|\lambda|^{2\theta n}$ holds, i.e.,

$$k_0(n) = \lfloor \frac{\log 2C + 2\theta n \log\frac{1}{|\lambda|}}{\log \mu_{max}} - 2n \rfloor < k_1(n). \tag{7.0.11}$$

Next we consider the case when $k \in [k_1(n), k_2(n)]$, where $k_2(n)$ is the largest integer such that

$$C\mu_{min}^{2n+k_2(n)}|\lambda|^{2\theta n} < 2, \tag{7.0.12}$$

i.e., we have

$$k_2(n) = \lfloor \frac{\log \frac{2}{C} + 2\theta n \log \frac{1}{|\lambda|}}{\log \mu_{min}} - 2n \rfloor. \tag{7.0.13}$$

Denote $t_{n,k} = (\frac{L}{C})^{\frac{1}{2n+k}}(\frac{1}{|\lambda|})^{\frac{2\theta n}{2n+k}}$, then we have

$$Ct_{n,k}^{2n+k}|\lambda|^{2\theta n} = L. \tag{7.0.14}$$

Now $K_{n,k}(t)$ is a holomorphic map from $T(\mu_{min}, t_{n,k})$ into $T(\frac{1}{C}, C)$.

Now consider $T(\mu_{min}, t_{n,k_2(n)})$, then all $K_{n,t}(t)$ is well defined on it for $k \leqslant k_2(n)$. When $n$ is large enough, we may take the following condition holds:

$$\frac{2}{\mu_{min}} M|\lambda|^{k_0(n)} < \frac{1}{2C^2} \tag{7.0.15}$$

for $k \in [k_0(n), k_2(n)]$. Let $\gamma$ be the circle $\{t||t| = t_0\}$ with $t_0 \in (\mu_{min}, t_{n,k_2(n)})$. Then for each $k \in [k_0(n), k_2(n) - 1]$, define the following map:

$$\begin{aligned} H_k : [0,1] \times \gamma &\longrightarrow \mathbb{C} \\ (s,t) &\longrightarrow (1-s)K_{n,k}(t) + sK_{n,k+1}(t). \end{aligned}$$

By inequality (7.0.8) and condition (7.0.15), we have

$$|H_k(s,t)| \leqslant |K_{n,k}(t)| + s|K_{n,k+1}(t) - K_{n,k}(t)| < 2C,$$

and

$$|H_k(s,t)| \geqslant |K_{n,k}(t)| - s|K_{n,k+1}(t) - K_{n,k}(t)| < \frac{1}{2C}.$$

Thus $H_k$ defines a homotopy from $K_{n,k}(\gamma)$ to $K_{n,k+1}(\gamma)$ inside $T(\frac{1}{2C}, 2C)$. As a consequence, $K_{n,k+1}(t)$ has the same degree with $K_{n,k}(t)$ as maps from $T(\mu_{min}, t_{n,k_2(n)})$ into $T(\frac{1}{2C}, 2C)$.

Thus when $n$ large enough, all $K_{n,k}(t)$ have the same degree for $k \in [k_0(n), k_2(n)]$, we may denote the degree as $d_n$. Then we have $d_n \in [-\frac{2\log C}{\log \frac{\mu_{max}}{\mu_{min}}}, \frac{2\log C}{\log \frac{\mu_{max}}{\mu_{min}}}]$. Then we know that

$\mathrm{sv}(k,\theta,n,t,a_n(t))$ is a holomorphic map from $T(\mu_{min},t_{n,k})$ into $T(\frac{1}{C}\mu_{min}^{2n+k}|\lambda|^{2\theta n},L)$ inside $D_y$ with degree $2n+k+d_n$. We may assume $L$ large enough such that $L>2C^2$, then $\frac{\log\frac{L}{2C^2}}{\log\frac{L}{2}}\in(0,1)$. Furthermore, the image contains an annuli $T(C\mu_{min}^{2n+k}|\lambda|^{2\theta n},\frac{L}{C^2})$ since $\frac{1}{C}t_{n,k}^{2n+k}|\lambda|^{2\theta n}=\frac{L}{C^2}>2$. We also have $q_2\in T(C\mu_{min}^{2n+k}|\lambda|^{2\theta n},\frac{L}{C^2})$.

Besides, by equation (7.0.7), we have

$$\frac{\partial\,\mathrm{sv}(k,\theta,n,t,a_n(t))}{\partial\mu}=(\frac{1}{K_{n,k}(t)}\frac{\partial K_{n,k}(t)}{\partial\mu}+(2n+k)\frac{1}{\mu})\,\mathrm{sv}(k,\theta,n,t,a_n(t)). \tag{7.0.16}$$

By replacing $T(\mu_{min},\mu_{max})$ by some interior $T(\mu_{min}+\delta,\mu_{max}-\delta)$ and take the interior as the new domain of $\mu$, we may assume $T(\mu_{min},\mu_{max})$ is compactly supported in $T(\hat{\mu}_{min},\hat{\mu}_{max})$ with

$$\hat{\mu}_{min}<\mu_{min}<\mu_{max}<\hat{\mu}_{max}. \tag{7.0.17}$$

Thus by Koebe's Distortion Theorem, we have $\frac{\partial K_{n,k}(t)}{\partial\mu}$ is uniformly bounded. Then when $n$ large enough, we have $\frac{\partial\,\mathrm{sv}(k,\theta,n,t,a_n(t))}{\partial\mu}$ is never zero. In conclusion, we have finished the proof. $\square$

By the meaning of equation (7.0.3) and the definition of $b_{n,k,i}(t)$ and $a_n(t)$, we have the following corollary:

**Corollary 7.0.3.** *For every $(\lambda,\tau)\in T_\lambda\times U$, when $k\in[k_0(n),k_2(n)]$, for each $i=1,2$, we have $a_n$ intersects with $b_{n,k,i}$ transversally at finite many points, in total, there are $2n+k+d_n$ intersection points.*

Next we prove the following proposition:

**Proposition 7.0.4.** *For every given $(\lambda,\tau)\in T_\lambda\times U$, denote the solution set of equation (7.0.3) for $(n,k)$ as $\Gamma_{n,k}(\lambda,\tau)$. For any $t_0\in T(\mu_{min},\mu_{max})$, we have a sequence $t_n\in\Gamma_{n,k_n}$ such that*

$$\lim_n t_n=t_0$$

*with*

$$|t_n-t_0|=O(\frac{1}{n}), \tag{7.0.18}$$

*where*

$$\lim_n \frac{k_n}{n} = 2\theta \frac{\log \frac{1}{|\lambda|}}{\log |t_0|} - 2.$$

*Denote*

$$Lim\Gamma_{n,k_n} := \{\lim_n t_n | t_n \in \Gamma_{n,k_n}\}, \tag{7.0.19}$$

*then we have*

$$Lim\Gamma_{n,k_n} = \{t = |t_0|\}. \tag{7.0.20}$$

*For any arc $I$ in $\{t = |t_0|\}$, let $J_n := \{rz | r \in [1 - \frac{1}{n}, 1 + \frac{1}{n}], z \in I\}$. Then we have*

$$\lim_n \frac{\#\Gamma_{n,k_n} \cap J_n}{\#\Gamma_{n,k_n}} = \frac{|I|}{2\pi |t_0|}, \tag{7.0.21}$$

*where $|I|$ is the arc-length of $I$.*

First we prove the following lemma:

**Lemma 7.0.5.** *Let $T_i$ be the annuli $\{z | 0 < r_i < |z| < R_i\}$, i=1,2. Then a holomorphic map $f$ from $T_1$ into $T_2$ can be written as:*

$$f(z) = z^d e^{g(z)} \tag{7.0.22}$$

*on $T_1$, where $d$ is the degree of $f$ and $g(z)$ is a holomorphic function on $T_1$. Besides, we have*

$$\log r_2 - d \log R_1 < Re(g(z)) < \log R_2 - d \log r_1 \tag{7.0.23}$$

*for all $z \in T_1$.*

*Proof.* By theorem 7.0.1, we know $f$ has a bounded degree. Then for any circle $\Gamma$ around 0 contained in $T_1$, we have

$$\int_\Gamma \frac{f'}{f} dz = 2\pi i d. \tag{7.0.24}$$

Now let $h(z) = z^{-d} f(z)$, then we have

$$\int_\Gamma \frac{h'}{h} dz = 0. \tag{7.0.25}$$

Fix a point $a$ in $T_1$, then

$$g(z) = \int_a^z \frac{h'}{h} dz$$

is a well-defined holomorphic function on $T_1$. Thus $\frac{h'(z)}{h(z)} = g'(z)$, i.e., $(h \cdot e^{-g})' = 0$. We have

$$h(z) = ce^{g(z)}$$

where $c$ is some constant. We can choose another holomorphic function $g(z)$ such that

$$h(z) = e^{g(z)}.$$

Thus

$$f(z) = z^d e^{g(z)}.$$

Furthermore, since $f(z) \in T_2$, we have

$$r_2 < |z|^d e^{\mathrm{Re}(g(z))} < R_2$$

for all $z \in T_1$. Thus we have

$$\log r_2 - d \log R_1 < \mathrm{Re}(g(z)) < \log R_2 - d \log r_1$$

for all $z \in T_1$. □

Now we prove the proposition 7.0.4

*Proof.* By lemma 7.0.5, for each $K_{n,k}(t)$, we could decompose as

$$K_{n,k}(t) = \mu^{d_n} \exp(P_{n,k}(t)) \tag{7.0.26}$$

where $P_{n,k}(t)$ is holomorphic and $|Re(P_{n,k}(t))|$ is uniformly bounded:

$$\log \frac{1}{C} - d_n log\mu_{max} < Re(P_{n,k}(t)) < \log C - d_n \log \mu_{min}. \tag{7.0.27}$$

Besides, we have

$$\frac{\partial K_{n,k}(t)}{\partial \mu} = \mu^{d_n - 1} \exp(P_{n,k}(t))(d_n + \mu \cdot \frac{\partial P_{n,k}(t)}{\partial \mu}) = K_{n,k}(t)(\frac{d_n}{\mu} + \frac{\partial P_{n,k}(t)}{\partial \mu}). \tag{7.0.28}$$

Thus we have

$$\frac{\partial P_{n,k}(t)}{\partial \mu} = \frac{\frac{\partial K_{n,k}(t)}{\partial \mu}}{K_{n,k}(t)} - \frac{d_n}{t} \tag{7.0.29}$$

are uniformly bounded. Since

$$\exp(P_{n,k}(t)) = \exp(P_{n,k}(t) + 2\pi i), \tag{7.0.30}$$

we may assume

$$Im(P_{n,k}(t)) \in [0, CP] \tag{7.0.31}$$

for some positive constant $CP$. Now for each $n$, choose a $k_n \in [k_0(n), k_2(n)]$ such that

$$\lim_n \frac{k_n}{n} = 2\theta \frac{\log \frac{1}{|\lambda|}}{\log |t_0|} - 2, \tag{7.0.32}$$

then for each $\mu \in \Gamma_{n,k_n}$, we have

$$\exp(P_{n,k_n}(t))\mu^{2n+k_n+d_n}\lambda^{2\theta n} = q_2. \tag{7.0.33}$$

Thus it satisfies one of the following equations:

$$\exp(\frac{P_{n,k_n}(t)}{2n+k_n+d_n})\mu = \frac{(q_2)^{\frac{1}{2n+k_n+d_n}}}{(\lambda)^{\frac{2\theta n}{2n+k_n+d_n}}} \exp(\frac{2s\pi i}{2n+k_n+d_n}), \tag{7.0.34}$$

where $s = 0, 1, ..., 2n + k_n + d_n - 1$. Since we have

$$\frac{\partial}{\partial \mu}(\exp(\frac{P_{n,k_n}(t)}{2n+k_n+d_n})\mu) = \exp(\frac{P_{n,k_n}(t)}{2n+k_n+d_n})(1 + \frac{\mu \cdot \frac{\partial P_{n,k}(t)}{\partial \mu}}{2n+k_n+d_n}). \tag{7.0.35}$$

Then we have

$$\lim_n |\frac{\partial}{\partial \mu}(\exp(\frac{P_{n,k_n}(t)}{2n+k_n+d_n})\mu)| = 1. \tag{7.0.36}$$

When $n$ large enough, we may assume

$$|\frac{\partial}{\partial \mu}(\exp(\frac{P_{n,k_n}(t)}{2n+k_n+d_n})\mu)| > \frac{1}{2}. \tag{7.0.37}$$

Thus $\exp(\frac{P_{n,k_n}(t)}{2n+k_n+d_n})\mu$ is a local diffeomorphism when $n$ large enough, we conclude that equations ( 7.0.34 ) have exactly 1 solution for each $s = 0, ..., 2n + k_n + d_n - 1$. Since now we have

$$\lim_n \exp(\frac{P_{n,k_n}(t)}{2n+k_n+d_n}) = 1 \tag{7.0.38}$$

and

$$\lim_n q_2^{\frac{1}{2n+k_n+d_n}} = 1, \tag{7.0.39}$$

$$\lim_n (\frac{1}{\lambda})^{\frac{2\theta n}{2n+k_n+d_n}} = (\frac{1}{\lambda})^{\frac{\log|t_0|}{\log\frac{1}{|\lambda|}}} = |t_0|(\frac{|\lambda|}{\lambda})^{\frac{\log|t_0|}{\log\frac{1}{|\lambda|}}}. \tag{7.0.40}$$

Denote $Arg(z)$ be the argument of complex number $z$ which in $[0, 2\pi)$. Then for $n$ large enough, choose $s_n$ such that

$$|\frac{2s_n\pi + Arg(q_2)}{2n+k_n+d_n} - Arg(\lambda)\frac{2\theta n}{2n+k_n+d_n} + \frac{CP}{2(2n+k_n+d_n)} - Arg(t_0)| \leqslant \frac{\pi}{2n+k_n+d_n}. \tag{7.0.41}$$

Let $t_n$ be the number in $\Gamma_{n,k_n}$ corresponding to $s_n$ in equation (7.0.34), then we have

$$||t_n| - |t_0|| = O(\frac{1}{n}) \tag{7.0.42}$$

and

$$|Arg(t_n) - Arg(t_0)| < \frac{CP + 2\pi}{2(2n+k_n+d_n)} \tag{7.0.43}$$

for $n$ large enough. Besides, for every $t_0^* \in \{t||t| = |t_0|\}$, we may choose $s_n^*$ such that condition (7.0.41) is satisfied for $s_n = s_n^*, t_0 = t_0^*$. Thus there exist $t_n^* \in \Gamma_{n,k_n}$ such that

$$||t_n^*| - |t_0|| = O(\frac{1}{n}) \tag{7.0.44}$$

,

$$|Arg(t_n^*) - Arg(t_0^*)| < \frac{CP + 2\pi}{2(2n+k_n+d_n)} \tag{7.0.45}$$

and

$$|\frac{2(s_n - s_n^*)\pi}{2n+k_n+d_n}| < |Arg(t_0) - Arg(t_0^*)| + \frac{2\pi}{2(2n+k_n+d_n)} \tag{7.0.46}$$

for $n$ large enough. Thus we have

$$Lim\Gamma_{n,k_n} = \{t = |t_0|\}. \tag{7.0.47}$$

and for any arc $I$ in $\{t = |t_0|\}$, let $J_n := \{rz | r \in [1 - \frac{1}{n}, 1 + \frac{1}{n}], z \in I\}$. Then we have

$$\lim_n \frac{\#\Gamma_{n,k_n} \cap J_n}{\#\Gamma_{n,k_n}} = \frac{|I|}{2\pi|t_0|}, \tag{7.0.48}$$

where $|I|$ is the arc-length of $I$. $\square$

The next proposition is about extending the solutions of equation (7.0.3) through the $\lambda$ parameter. First of all, since integer-valued $d_n$ depends continuously on $\lambda$, thus $d_n$ would be constant over a open subset in $\lambda$ parameter. Next, for $n$ large enough, the solvability of equation (7.0.3) turns to $k \in [k_0(n), k_2(n)]$, where by equations (7.0.11, 7.0.13), the 2 integers depends on $\lambda$ and they are constants over a circle $\{|\lambda| = l\}$. Now we state the following lemma:

**Lemma 7.0.6.** *Suppose for a parameter pair $(\lambda, \mu)$, we can find some $(n_0, k_0)$ solve the equation (7.0.3), then there exist $l_1, l_2 \in (0,1)$, which satisfies*

$$l_1 = \frac{(2C)^{\frac{1}{2\theta n_0}}}{(\mu_{\max})^{\frac{k_0+2n_0}{2\theta n_0}}} \tag{7.0.49}$$

*and*

$$l_2 = \frac{\left(\frac{2}{C}\right)^{\frac{1}{2\theta n_0}}}{(\mu_{\min})^{\frac{k_0+2n_0}{2\theta n}}}, \tag{7.0.50}$$

*where $C$ is the constant in the formula (7.0.11, 7.0.13). Then consider the annuli defined by $T_\lambda(l_1, l_2) = \{\lambda | \max\{l_1, \lambda_{min}\} < |\lambda| < \min\{l_2, \lambda_{max}\}\}$. For every $\lambda \in T_\lambda(l_1, l_2)$, we can solve equation (7.0.3) for $(n_0, k_0)$. In other words, the solution can be extended to the annuli.*

*Proof.* We only need to consider the equations:

$$k_0(n_0) = k_0, k_2(n_0) = k_0. \tag{7.0.51}$$

By equations (7.0.11, 7.0.13), we can see $l_1, l_2$ are the solutions of above equations respectively.

□

Now we denote the $\Gamma_{n,k}$ as the solutions of equation (7.0.3) over $T_\mu \times T_\lambda \times U$,then we know, $\Gamma_{n,k} = \cup_{\lambda,\tau}\Gamma_{n,k}(\lambda, \tau)$, then by above lemma, we know the points in each slice can be extended. Thus $\Gamma_{n,k}$ has several disjoint components and each component can be viewed as a finite cover over the annuli $T_\mu$. Now we can extend proposition 7.0.4 to $T_\mu \times T_\lambda$, the Palis invariant will arise naturally:

**Proposition 7.0.7.** *Denote the solution set of equation (7.0.3) for $(n, k)$ as $\Gamma_{n,k}(\lambda, \tau)$. For any $\beta \in [\frac{1}{\theta_0}, \frac{3}{2\theta_1}]$, we have a sequence $k_n$ with*

$$\lim_{n} \frac{k_n}{n} = 2\theta\beta - 2.$$

*Denote*

$$Lim\Gamma_{n,k_n} := \{(\mu, \lambda) | (\mu, \lambda) = \lim_{n} (\mu_n, \lambda_n) \text{ where } (\mu_n, \lambda_{1,n}) \in \Gamma_{n,k_n}\}, \tag{7.0.52}$$

*then we know that $Lim\Gamma_{n,k_n}$ is the level set of the Palis invariant $\{Pa(f) = \beta\}$.*

# Chapter 8

# The Newhouse phenomenon

First of all we will construct the set of Newhouse phenomenon in the phase space with a fixed $\lambda$ and $\tau$.

Now we have the following technical lemma:

**Lemma 8.0.1.** *Let $F$ be an unfolding of strong homoclinic tangency with respect to $(P, b)$, where $P = CY(b; \delta)$. Then there exist an integer $n_0 > 0$, such that the followings can be created:*

*(1). Holomorphic mappings $a_n : T(\mu_{min}, \mu_{max}) \longrightarrow \mathbb{C}$ with $Gr(a_n) \subset CY(b; \delta)$, such that there exist an uniform constant $\eta > 0$, such that when $n > n_0$, we have $CY(a_n; \frac{\eta}{\mu_{max}^{2n}}) \subset CY(b; \delta)$, and $F_{t,a}$ has a sink of period greater than $n$ for $a$ in $CY(a_n; \frac{\eta}{\mu_{max}^{2n}})$;*

*(2). Holomorphic mappings $b_{n,k,i} : T(\mu_{min}, \mu_{max}) \longrightarrow \mathbb{C}$, where $k$ in $\{k_0(n), \ldots, k_2(n)\}$, $i = 1, 2$, $Gr(b_{n,k,i}) \subset CY(b; \delta) \setminus Gr(b)$, such that $F_{t,b_{n,k,i}(t)}$ have a holomorphic quadratic homoclinic tangencies. We have*

$$0 < k_0(n) < k_2(n) < n, k_2(n) - k_0(n) = O(n). \tag{8.0.1}$$

*There exist an uniform constant $C > 0$, such that when $n > n_0$,*

$$\frac{1}{C|t|^n} < |b_{n,k,i}(t) - b(t)| < \frac{C}{|t|^n}, \tag{8.0.2}$$

*where* $t \in T(\mu_{min}, \mu_{max})$.

*Furthermore, there exists number* $\delta'_{n,k,i} > 0$ *such that* $F$ *is an unfolding of strong homoclinic tangency with respect to* $(P_{b_{n,k,i}}, b_{n,k,i})$, *where* $P_{b_{n,k,i}} := CY(b_{n,k,i}; \delta'_{n,k,i}) \subset CY(b; \delta)$. *Let us define* $\mathfrak{PB}((P,b))$ *to be the collection of all such pairs* $((P_{b_{n,k,i}}, b_{n,k,i}))$. *Let us denote*

$$\mathfrak{B}((P,b)) := \bigcup_{(P',b') \in \mathfrak{PB}((P,b))} Gr(b'). \tag{8.0.3}$$

*We have*

$$\overline{\mathfrak{B}((P,b))} = \mathfrak{B}((P,b)) \cup Gr(b); \tag{8.0.4}$$

*(3). For any* $a_n$, $k$ *in* $\{k_0(n), \ldots, k_2(n)\}$, $Gr(a_n)$ *intersects with the* 2 *graph of mappings* $Gr(b_{n,k,i})$, *where* $i = 1, 2$, *at, in total,* $2n + k + O(1)$ *points transversally. Let* $\Gamma_{b_{n,k,i}}$ *be the collection of* $t$*-coordinates of the intersection points of* $a_n$ *and* $b_{n,k,i}$, $\Gamma_{n,k}$ *be the union of* $\Gamma_{b_{n,k,i}}$ *for* $i = 1, 2$. *We know:*

$$\Gamma_{b_{n,k,1}} \cap \Gamma_{b_{n,k,2}} = \emptyset,$$

$$\#\Gamma_{b_{n,k,1}}, \#\Gamma_{b_{n,k,2}} > 0$$

$$\#\Gamma_{b_{n,k,1}} + \#\Gamma_{b_{n,k,2}} = 2n + k + O(1);$$

*(4).* $C\Gamma((P,b)) := \bigcup_{(P',b') \in \mathfrak{PB}((P,b))} \Gamma_{b'}$, *a dense subset of* $T(\mu_{min}, \mu_{max})$;

*(5). For any* $(P', b') \in \mathfrak{PB}((P,b))$, *without loss of generality, suppose* $b' = b_{n,k,1}$, *there exists* $\delta_{b'} > 0$ *such that, for any* $t \in \Gamma_{b'}$, *there exist an neighborhood* $U_{b'}(t)$ *of* $t$, *such that*

$$CY(b', U_{b'}(t); \delta_{b'}) \subset CY(a_n; \frac{\eta}{\mu_{max}^{2n}}) \cap P' \tag{8.0.5}$$

*and*

$$CY(b', U_{b'}(t); \delta_{b'}) \cap CY(b', U_{b'}(t'); \delta_{b'}) = \emptyset \tag{8.0.6}$$

*whenever* $t \neq t'$. *We call* $CY(b', U_{b'}(t); \delta_{b'})$ *a* ***Newhouse box of*** $((P,b),(P',b'))$ ***type*** *and let us denote* $INH((P',b')) := \bigcup_{t \in \Gamma_{b'}} CY(b', U_{b'}(t); \delta_{b'})$

*(6).*

$$G\Gamma((P,b)) := \bigcup_{(P',b')\in\mathfrak{PB}((P,b))} \bigcup_{t\in\Gamma_{b'}} U_{b'}(t), \tag{8.0.7}$$

$$NH((P,b)) := \bigcup_{(P',b')\in\mathfrak{PB}((P,b))} INH((P',b')); \tag{8.0.8}$$

*We have*

$$\pi_t NH((P,b)) = G\Gamma((P,b)), \tag{8.0.9}$$

$$C\Gamma((P,b)) \subset G\Gamma((P,b)), \tag{8.0.10}$$

$G\Gamma((P,b))$ *is an open and dense subset of* $T(\mu_{min},\mu_{max})$*, and*

$$\overline{NH((P,b))} \supset Gr(b). \tag{8.0.11}$$

*Proof.* By the definition of unfolding of strong homoclinic tangency with respect to $(P,b)$, it is enough to prove the lemma under the case $b(t)=0$ for all $t\in T(\mu_{min},\mu_{max})$. Part (1) follows from proposition 4.0.1;
Part (2) and inequality (8.0.2) follows from lemma 5.1.4 and definition 5.1.1. (8.0.4) follows from (8.0.2).
Part (3) follows from proposition 7.0.2.
Part (4) follows from proposition 7.0.4.
Part (5) follows from corollary 7.0.3.
Part (6), by the definition of $G\Gamma((P,b))$ and $NH((P,b))$, we know that (8.0.9) and (8.0.10) holds. As a consequence of $C\Gamma((P,b))$ is dense in $T(\mu_{min},\mu_{max})$, we know $G\Gamma((P,b))$ is an open and dense subset of $T(\mu_{min},\mu_{max})$. By the openness and denseness of $G\Gamma((P,b))$ and (8.0.4), we know (8.0.11) holds. □

By lemma 8.0.1, we can define a tree as follows:

**Definition 8.0.1.** Let $F$ be an unfolding of strong homoclinic tangency with respect to $(P,b)$, where $P=CY(b;\delta)$. Define a tree $Tree((P,b))$ as follows inductively on the level of the tree:
(1). The root node is (P,b);

(2). For any node $(P', b')$ of level $k$ in the tree, define its child nodes are all the elements from $\mathfrak{PB}((P', b'))$, where $\mathfrak{PB}((P', b'))$ is defined in (2) part of lemma 8.0.1.

Then we can define the following operations on each node of the tree:

For any node $((P', b')) \in Tree((P, b))$, denote $l((P', b'))$ be the level of this node. By lemma 8.0.1, we can also define $C\Gamma((P', b'))$, $G\Gamma((P', b'))$ and $NH((P', b'))$. When $l((P', b')) = l$, for any $(P'', b'') \in \mathfrak{PB}((P', b'))$, we call the Newhouse boxes of $((P', b'), (P'', b''))$ type the **Newhouse boxes of** $((P', b'), (P'', b''))$ **type in** $l$**-th generation**. For every natural number $l$, denote the following:

$$NH^{(l)}((P, b)) := \bigcup_{(P', b') \in Tree((P, b)), l((P', b')) = l} NH((P', b')). \tag{8.0.12}$$

Finally, we define the Newhouse set of the tree as follows:

$$\mathfrak{NH}((P, b)) := \bigcap_{n} \bigcup_{l \geqslant n} NH^{(l)}((P, b)). \tag{8.0.13}$$

We also denote $\mathfrak{B}((P, b))$ to be the following:

$$\mathfrak{B}((P, b)) = \bigcup_{(P', b') \in Tree((P, b))} Gr(b') \tag{8.0.14}$$

For the Newhouse boxes of any generation, we have the following lemma:

**Lemma 8.0.2.** *(1). Let $(P', b') \in Tree((P, b))$ of level $l$, let $U$ be a Newhouse box of $((P', b'), (P'', b''))$ type in $l$-th generation. Then we know the following:*

*$U \cap NH((P'', b''))$ contains infinitely many Newhouse boxes in $l+1$-th generation.*

*(2). $\bigcap_{l \geqslant 0} NH^{(l)}((P, b))$ is nonempty and*

$$\overline{\bigcap_{l \geqslant 0} NH^{(l)}((P, b))} \supset Gr(b). \tag{8.0.15}$$

*Proof.* (1).For any point $(t, b'(t)) \in U \cap Gr(b'')$, using the denseness of $C\Gamma((P'', b''))$ and inequality (8.0.2) in lemma 8.0.1, we know there exists an infinity sequence $\{n_k\}$, a sequence of nodes $\{(P_{n_k}, b_{n_k})\}$ where $(P_{n_k}, b_{n_k}) \in \mathfrak{PB}((P'', b''))$, and $t_{n_k} \in \Gamma_{b_{n_k}}$, such that

$$\lim_{k \to \infty} (t_{n_k}, b_{n_k}(t_{n_k})) = (t, b''(t)). \tag{8.0.16}$$

Thus when $k$ large enough, we can find infinitely many disjoint Newhouse boxes in $l+1$-th generation containing $(t_{n_k}, b_{n_k}(t_{n_k}))$ inside $U$.

(2). Let $U$ be a Newhouse box in 0-th generation. By part (1), we know that $U$ contains infinitely many Newhouse boxes in first generation. By repeatedly using part (1), we know that $U$ contains infinitely many chains of nested Newhouse boxes in all generation. Since $NH((P,b))$ is the union of all the Newhouse boxes of 0-th generation, and $\overline{NH((P,b))} \supset Gr(b)$, we know that

$$\overline{\bigcap_{l \geqslant 0} NH^{(l)}((P,b))} \supset Gr(b). \tag{8.0.17}$$

□

Now we state a theorem of the set of Newhouse phenomenon and give the proof.

**Theorem 8.0.3.** *For any given $\lambda$, denote $(P,b)$ to be the pair $(T(\mu_{min}, \mu_{max}) \times \mathbb{D}_a(r), O)$. Then there exists a set $\mathfrak{NH}((P,b)) \subset P$ such that $F_{t,a}$ have infinitely many sinks for each $(t,a) \in \mathfrak{NH}((P,b))$. Furthermore, $\overline{\mathfrak{NH}((P,b))} = \overline{\mathfrak{B}((P,b))}$.*

*Proof.* From lemma 8.0.1, definition 8.0.1 and lemma 8.0.2, we only need to prove $\overline{\mathfrak{NH}((P,b))} = \overline{\mathfrak{B}((P,b))}$.

First we prove $\overline{\mathfrak{NH}((P,b))}$ contains $\overline{\mathfrak{B}((P,b))}$.

For any node $(P', b') \in Tree((P,b))$, we know $Tree((P',b'))$ is just the sub-tree of $Tree((P,b))$ with root $(P', b')$, then by lemma 8.0.2, we know that

$$\overline{\bigcap_{l \geqslant 0} NH^{(l)}((P',b'))} \supset Gr(b'). \tag{8.0.18}$$

Thus we know

$$\overline{\mathfrak{NH}((P,b))} \supset \overline{\bigcap_{l \geqslant 0} NH^{(l)}((P',b'))} \supset Gr(b'). \tag{8.0.19}$$

This proves $\overline{\mathfrak{NH}((P,b))}$ contains $\overline{\mathfrak{B}((P,b))}$.

Then we prove $\overline{\mathfrak{NH}((P,b))} \subset \overline{\mathfrak{B}((P,b))}$.

For any point $z \in \overline{\mathfrak{NH}((P,b))}$, we have a sequence of points $\{z_n\}$ converging to $z$ with $z_n \in \mathfrak{NH}((P,b))$, thus for any $z_n$, there exist a infinity strictly-increasing sequences of positive integers $I_n$, such that $z_n \in \bigcap_{l \in I_n} NH^{(l)}((P,b))$. Thus when $n$ large enough, for each $z_n$, we can find a point $\widetilde{z}_n$ such that $||z_n - \widetilde{z}_n|| < \frac{1}{n}$, and there exist a node $(P_n, b_n)$, such that $\widetilde{z}_n \in Gr(b_n)$. Thus we have

$$\lim_n \widetilde{z}_n = z \tag{8.0.20}$$

and

$$\lim_n \widetilde{z}_n \in \overline{\mathfrak{B}((P,b))}. \tag{8.0.21}$$

This proves $\overline{\mathfrak{NH}((P,b))} \subset \overline{\mathfrak{B}((P,b))}$. Overall, we have

$$\overline{\mathfrak{NH}((P,b))} = \overline{\mathfrak{B}((P,b))}. \tag{8.0.22}$$

□

Furthermore, the set of total Newhouse phenomenon $\mathfrak{NH}$ would be the union of all such $\mathfrak{NH}((P,b))$ with $(\lambda, \tau)$ moving all over the $T_\lambda \times U$.

## 8.1 Newhouse Phenomenon and the lamination of Collect-Eckmann condition

Now we have constructed 2 objects of the Collect-Eckmann condition and the Newhouse Phenomenon, denoted as $\mathfrak{CE}$ and $\mathfrak{NH}$ respectively, in previous sections. We now want to consider their relationships. First of all, it is easy to see the following fact

**Proposition 8.1.1.** $\mathfrak{NH} \subset \mathfrak{CE}$.

We next state the following proposition:

**Proposition 8.1.2.** *There exist leaves of $\mathfrak{CE}$ such that for every leaf, there exist a dense subset of that leaf consisting of Newhouse points.*

*Proof.* We take a countable basis $\mathfrak{B}$ of $T_\lambda \times T_\mu$, label them by $B_i$. The Palis invariant gives a foliation of this product, and the range of Palis invariant is $[\frac{1}{\theta_0}, \frac{3}{2\theta_1}]$, now we choose a countable dense subset $\{\beta_i\}$ of the interval $[\frac{1}{\theta_0}, \frac{3}{2\theta_1}]$. Now we construct the desired leaf in $\mathfrak{CE}$ as follows: First, we give pairs of positive integers $(i, j)$ in lexicographical order, then we construct the renormalization scheme as follows: For each pair $(i, j)$, suppose for $(i, j-1)$, we have a open subset of Newhouse boxes $NH_{i,j-1}$ inside $B_i$, choose 2 small balls $D_1 \subsetneq D_2$ inside $NH_{i,j-1}$, we can find a denseness of $\{\beta_i\}$, there exist some $\beta$ such that the level set $\{Pa(f) = \beta\}$ intersect with $D_1$ transversally, thus by lemma 7.0.7, we can find a some $(n, k)$ such that global sink box with this $(n, k)$ intersect with $D_2$ transversally, thus we denote this open subset of $NH_{i,j-1}$ as $NH_{i,j}$.

Finally, by construction, this renormalization process will give a leaf in $\mathfrak{CE}$, and then for each $B_i$, by considering the intersection of $NH_{i,j}$ for all $j$, this will give a Newhouse point with projects inside $B_i$. Since $B_i$ gives a basis of $T_\lambda \times T_\mu$. We prove our conclusion. □

## 8.2 Proof of Theorem B

Now we give a proof of Theorem B.

*Proof.* The first 2 part of the theorem are proved in Theorem 8.0.3, Proposition 8.1.1 and 8.1.2. Now we prove part (3).

Let $\sigma \in \mathfrak{NH}$, $P_\sigma^{(l)}$ be the a sink created in the $l$-the level of renormalization scheme and $O(P_\sigma^{(l)})$ be the orbit of the sink, and let $c_\sigma^{(l)}$ be the point with finite time Collet-Eckmann condition created in lemma 6.2.1, which leads to

$$\lim_{l \longrightarrow \infty} c_\sigma^{(l)} = c(\sigma). \tag{8.2.1}$$

Then we have the following:

$$\text{dist}(c_\sigma^{(l)}, O(P_\sigma^{(l+1)})) \longrightarrow 0, \text{ when } n^{(l+1)} \longrightarrow \infty, \tag{8.2.2}$$

where dist stands for the Hausdorff distance between sets. Furthermore, if we denote $O^T(p)$ be the forward orbit of point $p$ from time 0 to time $T$, then we have the following:

$$\text{dist}_{\text{D}}(O^{T^{(l)}}(c_\sigma^{(l)}), O^{T^{(l)}}(c(\sigma)) \longrightarrow 0, \text{ when } l \longrightarrow \infty, \tag{8.2.3}$$

where $\mathrm{dist}_{\mathrm{D}}$ denotes the dynamical distance of two orbits, i.e.,

$$\mathrm{dist}_{\mathrm{D}}(O^n(P), O^n(Q)) := \sup_{0 \leqslant i \leqslant n} \mathrm{dist}(F^i(P), F^i(Q)). \tag{8.2.4}$$

Then we have

$$\omega(c(\sigma)) = \overline{\cup_l O(P_\sigma^{(l)})} \setminus \cup_l O(P_\sigma^{(l)}). \tag{8.2.5}$$

$\square$

# Chapter 9

# Appendix

Denote $B^s(D_x^{m-1}(100), D)$ be the space of bounded holomorphic maps from $D_x^{m-1}(100)$ into $D$, such that, $\forall f \in B^s(D_x^{m-1}(100), D)$, it satisfies the following two properties:
(A) it can be represented as a graph of a holomorphic function g which maps from from $D_x^{m-1}(100)$ into $D_y(100)$, i.e., $f(x) = (x, g(x))$, $\forall x \in D_x^{m-1}(100)$;
(B)$\forall x \in D_x^{m-1}(100)$, $Df_x(T_x D_x^{m-1}) \subseteq C^s_{f(x)}$.
We endow $B^s(D_x^{m-1}(100), D)$ with supremum norm for any two maps $f, g$ in $B^s(D_x^{m-1}(100), D)$:

$$d(f,g) = \underset{\gamma}{sup}\{ \text{ length of intersection of } f, g \text{ by } \gamma \text{ using restricted metric on } \gamma\},$$

where $\gamma$ run over all almost vertical curves in $D$.

Define $K^s_{t,a} := i_D \circ F^{-1}_{t,a}$ where $i_D$ is the restriction onto $D$, actually $K^s_{t,a}$ is just the graph transformation. Thus we have $d(K^s_{t,a}(f), K^s_{t,a}(g)) \leqslant \frac{1}{|\mu|-\epsilon} d(f,g)$

Denote $D_x^{m-1}(100) \times \{0\}$ by $\mathbb{O}$, then $\mathbb{O}$ is the unique attracting fixed point for the action $K^s_{t,a}$ on the metric space $(B^s(D_x^{m-1}(100), D), d)$.

Thus $d(K^s_{t,a}(f), \mathbb{O}) \leqslant \frac{1}{|\mu|-\epsilon} d(f, \mathbb{O})$. Then denote $f_i := (K^s_{t,a})^i(f)$, we have $d(f_i, \mathbb{O}) \leqslant \frac{1}{(|\mu|-\epsilon)^i} d(f, \mathbb{O})$.

Now differentiate $f_n = K^s_{t,a}(f_{n-1})$ by $a$, we have

$$\frac{\partial f_n}{\partial a} = \frac{\partial K^s_{t,a}}{\partial f}(f_{n-1}) \frac{\partial f_{n-1}}{\partial a} + \frac{\partial K^s_{t,a}}{\partial a}(f_{n-1}).$$

Apply above equation repeatedly, we have

$$\frac{\partial f_n}{\partial a} = \frac{\partial K^s_{t,a}}{\partial f}(f_{n-1}) \cdots \frac{\partial K^s_{t,a}}{\partial f}(f_k)\frac{\partial f_k}{\partial a} + \sum_{i=k}^{n-1}(\prod_{j=k+1}^{n-1} \frac{\partial K^s_{t,a}}{\partial f}(f_j))\frac{\partial K^s_{t,a}}{\partial a}(f_i)$$

for any $1 \leqslant k \leqslant n-1$.

Since $\|\frac{\partial K^s_{t,a}}{\partial f}\| \leqslant \frac{1}{|\mu|-\epsilon}$, $\|\frac{\partial K^s_{t,a}}{\partial a}(\mathbb{O})\| = 0$. we have $\|\frac{\partial K^s_{t,a}}{\partial a}(f)\| \leqslant L\,\mathrm{d}(f, \mathbb{O})$ when $\|f\|$ is small enough and $L$ depends uniformly on $\|f\|$.

Thus we have the following lemma:

**Lemma 9.0.1.**

$$\|\frac{\partial f_n}{\partial a}\| = O(\frac{n}{(|\mu| - \epsilon)^n}).$$

# Bibliography


[AM03] Artur Avila and Carlos Gustavo Moreira. "Statistical properties of unimodal maps: smooth families with negative Schwarzian derivative". In: 286. Geometric methods in dynamics. I. 2003, pp. xviii, 81–118.

[AM05] Artur Avila and Carlos Gustavo Moreira. "Statistical properties of unimodal maps: the quadratic family". In: *Ann. of Math. (2)* 161.2 (2005), pp. 831–881. ISSN: 0003-486X. DOI: 10.4007/annals.2005.161.831. URL: https://doi.org/10.4007/annals.2005.161.831.

[Ano67] D. V. Anosov. "Geodesic flows on closed Riemannian manifolds of negative curvature". In: *Trudy Mat. Inst. Steklov.* 90 (1967), p. 209. ISSN: 0371-9685.

[AP37] A. Andronov and L. S. Pontryagin. "Systèmes grossiers". French. In: *C. R. (Dokl.) Acad. Sci. URSS, n. Ser.* 14 (1937), pp. 247–250. ISSN: 1819-0723.

[AŠ73] V. S. Afraımovič and L. P. Šil nikov. "The singular sets of Morse-Smale systems". In: *Trudy Moskov. Mat. Obšč.* 28 (1973), pp. 181–214. ISSN: 0134-8663.

[Asp04] Magnus Aspenberg. *The Collet-Eckmann condition for rational functions on the Riemann sphere*. Thesis (Tekn.dr.)–Kungliga Tekniska Hogskolan (Sweden). ProQuest LLC, Ann Arbor, MI, 2004, p. 70. ISBN: 978-9172-83809-3. URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:C819791.

[Ast+19] Matthieu Astorg et al. "Collet, Eckmann and the bifurcation measure". In: *Invent. Math.* 217.3 (2019), pp. 749–797. ISSN: 0020-9910. DOI: 10.1007/s00222-019-00874-5. URL: https://doi.org/10.1007/s00222-019-00874-5.

[BC85] Michael Benedicks and Lennart Carleson. "On iterations of $1-ax^2$ on $(-1,1)$". In: *Ann. of Math. (2)* 122.1 (1985), pp. 1–25. ISSN: 0003-486X. DOI: 10.2307/1971367. URL: https://doi.org/10.2307/1971367.

[BC91] Michael Benedicks and Lennart Carleson. "The dynamics of the Hénon map". In: *Ann. of Math. (2)* 133.1 (1991), pp. 73–169. ISSN: 0003-486X. DOI: 10.2307/2944326. URL: https://doi.org/10.2307/2944326.

[Ber16] Pierre Berger. "Generic family with robustly infinitely many sinks". In: *Invent. Math.* 205.1 (2016), pp. 121–172. ISSN: 0020-9910. DOI: 10.1007/s00222-015-0632-6. URL: https://doi.org/10.1007/s00222-015-0632-6.

[Ber18] Pierre Berger. "Lectures on structural stability in dynamics". In: *Dynamical systems*. Vol. 115. Banach Center Publ. Polish Acad. Sci. Inst. Math., Warsaw, 2018, pp. 9–35.

[Bie20] Sébastien Biebler. "Newhouse phenomenon for automorphisms of low degree in C3". In: *Advances in Mathematics* 361 (2020). ISSN: 10902082. DOI: 10.1016/j.aim.2019.106952.

[BK94] I. U. Bronstein and A. Ya. Kopanskiı. *Smooth invariant manifolds and normal forms*. Vol. 7. World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises. World Scientific Publishing Co., Inc., River Edge, NJ, 1994, pp. xii+384. ISBN: 981-02-1572-X. DOI: 10.1142/9789812798749. URL: https://doi.org/10.1142/9789812798749.

[BL91] A. M. Blokh and M. Yu. Lyubich. "Measurable dynamics of $S$-unimodal maps of the interval". In: *Ann. Sci. École Norm. Sup. (4)* 24.5 (1991), pp. 545–573. ISSN: 0012-9593. URL: http://www.numdam.org/item?id=ASENS_1991_4_24_5_545_0.

[BMP18] Michael Benedicks, Marco Martens, and Liviana Palmisano. *Newhouse Laminations*. 2018. DOI: 10.48550/ARXIV.1811.00617. URL: https://arxiv.org/abs/1811.00617.

[Buz97] Gregery T. Buzzard. "Infinitely many periodic attractors for holomorphic maps of 2 variables". In: *Ann. of Math. (2)* 145.2 (1997), pp. 389–417. ISSN: 0003-486X. DOI: 10.2307/2951819. URL: https://doi.org/10.2307/2951819.

[CE83] P. Collet and J.-P. Eckmann. "Positive Liapunov exponents and absolute continuity for maps of the interval". In: *Ergodic Theory Dynam. Systems* 3.1 (1983), pp. 13–46. ISSN: 0143-3857. DOI: 10.1017/S0143385700001802. URL: https://doi.org/10.1017/S0143385700001802.

[Dev89] Robert L. Devaney. *An introduction to chaotic dynamical systems*. Second. Addison-Wesley Studies in Nonlinearity. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA, 1989, pp. xviii+336. ISBN: 0-201-13046-7.

[DH85] Adrien Douady and John Hamal Hubbard. "On the dynamics of polynomial-like mappings". In: *Ann. Sci. École Norm. Sup. (4)* 18.2 (1985), pp. 287–343. ISSN: 0012-9593. URL: http://www.numdam.org/item?id=ASENS_1985_4_18_2_287_0.

[DL15] Romain Dujardin and Mikhail Lyubich. "Stability and bifurcations for dissipative polynomial automorphisms of $\mathbb{C}^2$". In: *Invent. Math.* 200.2 (2015), pp. 439–511. ISSN: 0020-9910. DOI: 10.1007/s00222-014-0535-y. URL: https://doi.org/10.1007/s00222-014-0535-y.

[DLM05] A. De Carvalho, M. Lyubich, and M. Martens. "Renormalization in the Hénon family. I. Universality but non-rigidity". In: *J. Stat. Phys.* 121.5-6 (2005), pp. 611–669. ISSN: 0022-4715. DOI: 10.1007/s10955-005-8668-4. URL: https://doi.org/10.1007/s10955-005-8668-4.

[EH70] Clifford J. Earle and Richard S. Hamilton. "A fixed point theorem for holomorphic mappings". In: *Global Analysis (Proc. Sympos. Pure Math., Vol. XVI, Berkeley, Calif., 1968)*. Amer. Math. Soc., Providence, R.I., 1970, pp. 61–65.

[FM08] Edson de Faria and Welington de Melo. *Mathematical tools for one-dimensional dynamics*. Vol. 115. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2008, pp. viii+191. ISBN: 978-0-521-88861-5. DOI: 10.1017/CBO9780511755231. URL: https://doi.org/10.1017/CBO9780511755231.

[FS01] John Erik Fornæss and Nessim Sibony. "Some open problems in higher dimensional complex analysis and complex dynamics". In: *Publ. Mat.* 45.2 (2001), pp. 529–547. ISSN: 0214-1493. DOI: 10.5565/PUBLMAT_45201_11. URL: https://doi.org/10.5565/PUBLMAT_45201_11.

[Gav98] Estela Ana Gavosto. "Attracting basins in $P^2$". In: *J. Geom. Anal.* 8.3 (1998), pp. 433–440. ISSN: 1050-6926. DOI: 10.1007/BF02921795. URL: https://doi.org/10.1007/BF02921795.

[GH83] John Guckenheimer and Philip Holmes. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*. Vol. 42. Applied Mathematical Sciences. Springer-Verlag, New York, 1983, pp. xvi+453. ISBN: 0-387-90819-6. DOI: 10.1007/978-1-4612-1140-2. URL: https://doi.org/10.1007/978-1-4612-1140-2.

[GS98] Jacek Graczyk and Stas Smirnov. "Collet, Eckmann and Hölder". In: *Invent. Math.* 133.1 (1998), pp. 69–96. ISSN: 0020-9910. DOI: 10.1007/s002220050239. URL: https://doi.org/10.1007/s002220050239.

[Guc79] John Guckenheimer. "Sensitive dependence to initial conditions for one-dimensional maps". In: *Comm. Math. Phys.* 70.2 (1979), pp. 133–160. ISSN: 0010-3616. URL: http://projecteuclid.org/euclid.cmp/1103907294.

[Har03] Lawrence A. Harris. "Fixed points of holomorphic mappings for domains in Banach spaces". In: *Abstr. Appl. Anal.* 5 (2003), pp. 261–274. ISSN: 1085-3375. DOI: 10.1155/S1085337503205042. URL: https://doi.org/10.1155/S1085337503205042.

[Hub51] Heinz Huber. "Über analytische Abbildungen von Ringgebieten in Ringgebiete". In: *Compositio Math.* 9 (1951), pp. 161–168. ISSN: 0010-437X.

[IY91] Yu. S. Il yashenko and S. Yu. Yakovenko. "Finitely smooth normal forms of local families of diffeomorphisms and vector fields". In: *Uspekhi Mat. Nauk* 46.1(277) (1991), pp. 3–39, 240. ISSN: 0042-1316. DOI: 10.1070/RM1991v046n01ABEH002733. URL: https://doi.org/10.1070/RM1991v046n01ABEH002733.

[KH95] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*. Vol. 54. Encyclopedia of Mathematics and its Applications. With a supplementary chapter by Katok and Leonardo Mendoza. Cambridge University Press, Cambridge, 1995, pp. xviii+802. ISBN: 0-521-34187-6. DOI: 10.1017/CBO9780511809187. URL: https://doi.org/10.1017/CBO9780511809187.

[KN92] Gerhard Keller and Tomasz Nowicki. "Spectral theory, zeta functions and the distribution of periodic points for Collet-Eckmann maps". In: *Comm. Math. Phys.* 149.1 (1992), pp. 31–69. ISSN: 0010-3616. URL: http://projecteuclid.org/euclid.cmp/1104251138.

[Kob70] Shoshichi Kobayashi. *Hyperbolic manifolds and holomorphic mappings*. Vol. 2. Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 1970, pp. ix+148.

[Led81] François Ledrappier. "Some properties of absolutely continuous invariant measures on an interval". In: *Ergodic Theory Dynam. Systems* 1.1 (1981), pp. 77–93. ISSN: 0143-3857. DOI: 10.1017/s0143385700001176. URL: https://doi.org/10.1017/s0143385700001176.

[Lor63] Edward N. Lorenz. "Deterministic nonperiodic flow". In: *J. Atmospheric Sci.* 20.2 (1963), pp. 130–141. ISSN: 0022-4928. DOI: 10.1175/1520-0469(1963)020<0130:DNF>2.0.CO;2. URL: https://doi.org/10.1175/1520-0469(1963)020%3C0130:DNF%3E2.0.CO;2.

[Lyu02] Mikhail Lyubich. "Almost every real quadratic map is either regular or stochastic". In: *Ann. of Math. (2)* 156.1 (2002), pp. 1–78. ISSN: 0003-486X. DOI: 10.2307/3597183. URL: https://doi.org/10.2307/3597183.

[McM94] Curtis T. McMullen. *Complex dynamics and renormalization*. Vol. 135. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1994, pp. x+214. ISBN: 0-691-02982-2; 0-691-02981-4.

[Mil06] John Milnor. *Dynamics in one complex variable*. Third. Vol. 160. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2006, pp. viii+304. ISBN: 978-0-691-12488-9; 0-691-12488-4.

[Mos69] J. Moser. "On a theorem of Anosov". In: *J. Differential Equations* 5 (1969), pp. 411–440. ISSN: 0022-0396. DOI: 10.1016/0022-0396(69)90083-7. URL: https://doi.org/10.1016/0022-0396(69)90083-7.

[MPT20] Marco Martens, Liviana Palmisano, and Zhuang Tao. "Newhouse Laminations of polynomials on C2". In: *International Journal of Mathematics* 31.11 (2020). ISSN: 0129167X. DOI: 10.1142/S0129167X20500913.

[MS93] Welington de Melo and Sebastian van Strien. *One-dimensional dynamics*. Vol. 25. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1993, pp. xiv+605. ISBN: 3-540-56412-8. DOI: 10.1007/978-3-642-78043-1. URL: https://doi.org/10.1007/978-3-642-78043-1.

[MV93] Leonardo Mora and Marcelo Viana. "Abundance of strange attractors". In: *Acta Math.* 171.1 (1993), pp. 1–71. ISSN: 0001-5962. DOI: 10.1007/BF02392766. URL: https://doi.org/10.1007/BF02392766.

[New74] Sheldon E. Newhouse. "Diffeomorphisms with infinitely many sinks". In: *Topology* 13 (1974), pp. 9–18. ISSN: 0040-9383. DOI: 10.1016/0040-9383(74)90034-2. URL: https://doi.org/10.1016/0040-9383(74)90034-2.

[NPT83] S. Newhouse, J. Palis, and F. Takens. "Bifurcations and stability of families of diffeomorphisms". In: *Inst. Hautes Études Sci. Publ. Math.* 57 (1983), pp. 5–71. ISSN: 0073-8301. URL: http://www.numdam.org/item?id=PMIHES_1983__57__5_0.

[NS88] T. Nowicki and S. van Strien. "Absolutely continuous invariant measures for $C^2$ unimodal maps satisfying the Collet-Eckmann conditions". In: *Invent. Math.* 93.3 (1988), pp. 619–635. ISSN: 0020-9910. DOI: 10.1007/BF01410202. URL: https://doi.org/10.1007/BF01410202.

[Pal00] Jacob Palis. "A global view of dynamics and a conjecture on the denseness of finitude of attractors". In: 261. Géométrie complexe et systèmes dynamiques (Orsay, 1995). 2000, pp. xiii–xiv, 335–347.

[Pal19] Liviana Palmisano. *Coexistence of non-periodic attractors*. 2019. DOI: 10.48550/ARXIV.1903.01446. URL: https://arxiv.org/abs/1903.01446.

[Pal78] J. Palis. "A differentiable invariant of topological conjugacies and moduli of stability". In: *Dynamical systems, Vol. III—Warsaw*. 1978, 335–346. Astérisque, No. 51.

[Poi90] H. Poincaré. "On the three-body problem and the equations of dynamics". French. In: *Acta Math.* 13 (1890), pp. 1–270. ISSN: 0001-5962.

[Pom92] Ch. Pommerenke. *Boundary behaviour of conformal maps*. Vol. 299. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1992, pp. x+300. ISBN: 3-540-54751-7. DOI: 10.1007/978-3-662-02770-7. URL: https://doi.org/10.1007/978-3-662-02770-7.

[PP11] Mihai N. Pascu and Nicolae R. Pascu. "Neighbourhoods of univalent functions". In: *Bull. Aust. Math. Soc.* 83.2 (2011), pp. 210–219. ISSN: 0004-9727. DOI: 10.1017/S0004972710000468. URL: https://doi.org/10.1017/S0004972710000468.

[Prz98] Feliks Przytycki. "Iterations of holomorphic Collet-Eckmann maps: conformal and invariant measures. Appendix: on non-renormalizable quadratic polynomials". In: *Trans. Amer. Math. Soc.* 350.2 (1998), pp. 717–742. ISSN: 0002-9947. DOI: 10.1090/S0002-9947-98-01890-X. URL: https://doi.org/10.1090/S0002-9947-98-01890-X.

[PS00] Enrique R. Pujals and Martın Sambarino. "Homoclinic tangencies and hyperbolicity for surface diffeomorphisms". In: *Ann. of Math. (2)* 151.3 (2000), pp. 961–1023. ISSN: 0003-486X. DOI: 10.2307/121127. URL: https://doi.org/10.2307/121127.

[PS70] J. Palis and S. Smale. "Structural stability theorems". In: *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)*. Amer. Math. Soc., Providence, R.I., 1970, pp. 223–231.

[PT93] Jacob Palis and Floris Takens. *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*. Vol. 35. Cambridge Studies in Advanced Mathematics. Fractal dimensions and infinitely many attractors. Cambridge University Press, Cambridge, 1993, pp. x+234. ISBN: 0-521-39064-8.

[Ros69a] Paul C. Rosenbloom. "Perturbation of the zeros of analytic functions. I". In: *J. Approximation Theory* 2 (1969), pp. 111–126. ISSN: 0021-9045. DOI: 10.1016/0021-9045(69)90035-5. URL: https://doi.org/10.1016/0021-9045(69)90035-5.

[Ros69b] Paul C. Rosenbloom. "Perturbation of zeros of analytic functions. II". In: *J. Approximation Theory* 2 (1969), pp. 275–300. ISSN: 0021-9045. DOI: 10.1016/0021-9045(69)90024-0. URL: https://doi.org/10.1016/0021-9045(69)90024-0.

[Shu87] Michael Shub. *Global stability of dynamical systems*. With the collaboration of Albert Fathi and Rémi Langevin, Translated from the French by Joseph Christy. Springer-Verlag, New York, 1987, pp. xii+150. ISBN: 0-387-96295-6. DOI: 10.1007/978-1-4757-1947-5. URL: https://doi.org/10.1007/978-1-4757-1947-5.

[Sin78] David Singer. "Stable orbits and bifurcation of maps of the interval". In: *SIAM J. Appl. Math.* 35.2 (1978), pp. 260–267. ISSN: 0036-1399. DOI: 10.1137/0135020. URL: https://doi.org/10.1137/0135020.

[Sma63] S. Smale. "A structurally stable differentiable homeomorphism with an infinite number of periodic points". In: *Qualitative methods in the theory of non-linear vibrations (Proc. Internat. Sympos. Non-linear Vibrations, Vol. II, 1961)*. Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1963, pp. 365–366.

[Sma65] Stephen Smale. "Diffeomorphisms with many periodic points". In: *Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse)*. Princeton Univ. Press, Princeton, N.J., 1965, pp. 63–80.

[Sma67] S. Smale. "Differentiable dynamical systems". In: *Bull. Amer. Math. Soc.* 73 (1967), pp. 747–817. ISSN: 0002-9904. DOI: 10.1090/S0002-9904-1967-11798-1. URL: https://doi.org/10.1090/S0002-9904-1967-11798-1.

[Tak11] Hiroki Takahasi. "Abundance of non-uniform hyperbolicity in bifurcations of surface endomorphisms". In: *Tokyo J. Math.* 34.1 (2011), pp. 53–113. ISSN: 0387-3870. DOI: 10.3836/tjm/1313074445. URL: https://doi.org/10.3836/tjm/1313074445.

[Tsu93] Masato Tsujii. "Positive Lyapunov exponents in families of one-dimensional dynamical systems". In: *Invent. Math.* 111.1 (1993), pp. 113–137. ISSN: 0020-9910. DOI: 10.1007/BF01231282. URL: https://doi.org/10.1007/BF01231282.

[VL03] M. Viana and S. Lutstsatto. "Exclusions of parameter values in Hénon-type systems". In: *Uspekhi Mat. Nauk* 58.6(354) (2003), pp. 3–44. ISSN: 0042-1316. DOI: 10.1070/RM2003v058n06ABEH000674. URL: https://doi.org/10.1070/RM2003v058n06ABEH000674.

[WY01] Qiudong Wang and Lai-Sang Young. "Strange attractors with one direction of instability". In: *Comm. Math. Phys.* 218.1 (2001), pp. 1–97. ISSN: 0010-3616. DOI: 10.1007/s002200100379. URL: https://doi.org/10.1007/s002200100379.

[WY08] Qiudong Wang and Lai-Sang Young. "Toward a theory of rank one attractors". In: *Ann. of Math. (2)* 167.2 (2008), pp. 349–480. ISSN: 0003-486X. DOI: 10.4007/annals.2008.167.349. URL: https://doi.org/10.4007/annals.2008.167.349.

[You92] L.-S. Young. "Decay of correlations for certain quadratic maps". In: *Comm. Math. Phys.* 146.1 (1992), pp. 123–138. ISSN: 0010-3616. URL: http://projecteuclid.org/euclid.cmp/1104249979.