Distortion Structure for Renormalization

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1 Introduction

One of the many subfields of mathematics concerns itself with the study of *dynamical systems*. The original motivation for this branch comes from physics, since it was quickly noted after the discovery of calculus by Newton and Leibniz that systems of differential equations often governed the behavior of models from Newtonian mechanics. Originally, mathematicians searched for exact solutions of these equations in the hopes of solving questions such as those concerning the stability of the Solar System. This changed in the late 19th century due to the efforts of Poincaré. What he understood was that it was more effective to try and understand qualitative properties of a dynamical system such as asymptotics or existence of periodic orbits rather than solve them explicitly. For his work on celestial mechanics concerning the 3-body problem, which he showed was fundamentally chaotic, he received King Oscar of Sweden's prize in 1887.

The mathematical formulation of a dynamical system is based around a *phase space* U, which is usually metrizable, and a *time evolution law*, which can either be continuous in time, like the flows of differential equations studied in physical systems, or a discrete map $f : U \to U$. Discrete maps occur naturally when you take a flow on a higher dimensional surface and restrict it to lower dimensional cross sections. The class of discrete dynamical systems studied in this thesis will be that of circle homeomorphisms. Their simplest classification was due to Poincaré, who discovered an invariant known as the *rotation number*. If the rotation number is rational, orbits of points converge to periodic orbits, while maps with irrational rotation numbers are semiconjugate to rotations. Further developments of the theory sought to answer the question of when a circle map with an irrational rotation number is conjugate to a rigid rotation, and how smooth the conjugating map is. Denjoy proved that a C^2 diffeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$ with irrational rotation number is topologically conjugate to a rotation, and constructed examples of C^1 diffeomorphisms for which the conjugacy fails to hold. Arnold proved that even for smooth maps, the conjugacy need only be continuous, and Herman proved a general result, which, for example concludes that if f is C^3 and has rotation number ρ lying in the full measure set of *Diophantine* rotation numbers (those which colloquially are badly approximated by rational numbers), then it is C^1 conjugate to a rotation. An interpretation of such results is to say that at sufficiently small scales, these circle maps are indistinguishable from rotations.

The proofs of these classic theorems involve studying the behavior of high iterates of f over nested decreasing sequences of dynamically defined intervals. A major innovation of modern times has been to do this using *renormalization*.

The basic principles of renormalization are as follows: One begins with a dynamical system defined by the set U and the mapping $f : U \to U$. Then, a subset V of U is chosen so that points in V eventually return under iteration. Finally, the set V and the first return map $\tilde{f} : V \to V$ are rescaled in the hopes that one will obtain a new dynamical system of the same class as the original. If that is the case, we say (U, f) is *renormalizable* and by iterating this process, we consider renormalization as an operator on a space of dynamical systems.

Renormalization was first introduced by Tressier, Collet and Feigenbaum to study the dynamics of unimodal maps such as the logistic family $f_r : [0,1] \to [0,1],\ f_r(x) = rx(1-x)$ for $r \in [0,4]$. Those systems were experimentally observed to exhibit universal properties associated to period doubling bifurcations. Lanford used a computer assisted proof to first establish that the renormalization operator had a fixed point. Rigorous theoretical advances were due to Sullivan, who established contraction of renormalization to the fixed point, and McMullen, who constructed the unstable manifold of renormalization. Lyubich then completed the picture by proving the renormalization operator was hyperbolic. With this in mind, the interest in extending these techniques to the field of circle maps can be well understood.

There are various conventions for renormalizing circle maps. In this thesis, the chosen method is to consider them as branched interval maps. These can be obtained by lifting the circle to $[0,1]$ and taking appropriate images of circle homeomorphisms $f : \mathbb{S}^1 \to \mathbb{S}^1$ that are C^2 everywhere except for at one marked point and its preimage. For circle maps with aperiodic dynamics, this data can be encoded by two spatial coordinates $c, v \in (0,1)$ and two C^2 orientation preserving diffeomorphisms $F_-, F_+ : [0,1] \to [0,1]$. Using the spatial coordinates to define a first return interval I and then affinely rescaling I to $[0,1]$, one obtains a new renormalized circle map.

An extensive analysis of renormalization of circle maps has been done by Khanin and Sinai, and also by Stark. As anticipated by Herman, if f is C^3 with Diophantine rotation number, then the renormalizations of f converge to rigid rotations. More interestingly, it was shown by Khanin and Teplitsky that if one allowed the derivative of f to have two break points in a way quantified by a real invariant $K \neq 0$, referred to here as the *total distortion*, the renormalization operator is hyperbolic, and the renormalizations of f converge to an attractor homeomorphic to the product of $\mathbb{R}$ with a Cantor set, for which the corresponding branches F_-, F_+ at every point are Möbius. The main result of this thesis shows that the renormalization operator has additional internal structure when restricted to the space of Möbius circle maps.

In its classical form, the definitions of renormalization for unimodal and circle maps involve composition, which can be difficult to work with. Originally in the setting of unimodal maps, Martens approached this issue by extending renormalization to the larger space of *decompositions*. For our purposes, a decomposition can be thought of as a function $\underline{f}$ from the set of dyadic rationals $\mathcal{T}$ in $(0,1]$ into the space of C^2 orientation preserving diffeomorphisms of $[0,1]$, which can be viewed as a chain of diffeomorphisms equipped with a time order of composition that respects the structure of the dyadic rationals. As a substitute for composing, two decompositions $\underline{f}, \underline{g}$ can then be concatenated together into a new decomposition $\underline{f} \star \underline{g}$ such that

$$(f \star g)_\tau = \begin{cases} f_{2\tau} & \text{if } \tau \leq 1/2, \\ g_{2\tau-1} & \text{if } \tau > 1/2. \end{cases}$$

By concatenating rather than composing, intrinsic information about the behavior of a map at small scales can be preserved and the past history of a renormalized map can more easily be recovered.

The attractor for renormalization of C^3 circle maps with a break point lies in the space of Mobius circle maps, and these have additional properties associated to them that allows for fundamentally different analysis. The orientation preserving Möbius diffeomorphisms that map the interval $[0, 1]$ to itself form a one-parameter family, characterized by the real number $\omega(M) = \int_0^1 M''/M'$, which is referred to here as the *weight*. The Möbius diffeomorphisms $(M_\omega)_{\omega \in \mathbb{R}}$ form an additive group under composition, so we can fully understand a decomposition $\underline{M}$ of Möbius diffeomorphisms in terms of its weight decomposition $\underline{\omega}$, with $\omega_\tau = \omega(M_\tau)$. From $\underline{\omega}$, one can then construct an atomic signed Borel measure $\mu_{\underline{\omega}}$ on $(0, 1]$ by the definition $\mu_{\underline{\omega}}(\{\tau\}) = \omega_\tau$.

The action of renormalization reorders weight decompositions by cutting them up and squeezing them into smaller time intervals. A result of this is that the associated atomic measures converge to continuous measures. This noticeably differs from the situation for unimodal maps, where the decompositions bend and preserve distortion at small scales. This suggests a paradigm shift, which is explored in this thesis. The key idea is to replace the branches (M_-, M_+) of a Möbius circle map by signed measures (μ_-, μ_+), which we will refer to as *distortion measures*. For reasons which will become clear, we will endow these measures with the norm

$$||\mu|| = \sup_{\tau \in \mathcal{T}} |\mu(0, \tau]|.$$

The distortion measures generate a *Möbius flow* defined at dyadic timescales $(0, \tau]$ by transforming the unit interval by a Möbius diffeomorphism of weight $\mu(0, \tau]$. Due to the group structure of Möbius diffeomorphisms, this construction is well defined, and the norm $||\mu||$ encapsulates the displacement of orbits of the flow.

Given the data (μ_-, μ_+), one can apply the *canonical projection* $\pi(\mu_-, \mu_+) = (M_{\mu_-(0,1]}, M_{\mu_+(0,1]})$ to obtain the branches of a Möbius circle map. What is now needed is a means of renormalizing distortion measures in a way that commutes with the canonical projection. The formulas for doing this turn out to only require *zooms* and *concatenations*. Just like for decompositions, one can concatenate two Möbius flows into a single flow by applying one after the other in sequential time order. The zoom operator, on the other hand, is analogous to restricting a Möbius flow to a smaller subinterval $I \subseteq [0, 1]$ of orbits and then rescaling the corresponding flow back to unit size. At the level of distortion measures, one obtains from μ a new measure $Z_I \mu$ such that for $\tau \in \mathcal{T}$,

$$(Z_I \mu)(0, \tau] = \int_I \frac{M''_{\mu(0,\tau]}}{M'_{\mu(0,\tau]}}.$$

The crucial analytical tool for studying the renormalization of distortion measures is the Zoom Contraction Law. It states that if μ, ν are measures such that $||\mu||, ||\nu|| < L$, and I is a subinterval of $[0,1]$ with $|I| < 1$, then there exists $\kappa < 1$ depending only on L and $|I|$ such that

$$||Z_I\mu - Z_I\nu|| < \kappa||\mu - \nu||.$$

The renormalization operators $\mathcal{R}_z^\sigma$ are quantified by a real number $\sigma \in (0,1)$, referred to as the *scaling ratio*, and $z \in \{-,+\}$ which selects the branch μ_z to be acted upon by a zoom. One can then show that any sequence of renormalizations defined by a *renormalization scheme* $(\underline{\sigma}, \underline{z}) \in (0,1)^{\mathbb{N}} \times \{-,+\}^{\mathbb{N}}$ that satisfies a mild combinatorial condition referred to as *δ-boundedness* contracts distances between distortion measure pairs $(\mu_-, \mu_+), (\nu_-, \nu_+)$ for which $\pi(\mu_-, \mu_+) = \pi(\nu_-, \nu_+)$. The essence of the proof goes as follows:
Firstly, the renormalization operator $\mathcal{R}_{z_n}^{\sigma_n}$ acts on the space $\mathcal{M} \times \mathcal{M}$ of distortion measure pairs by cutting μ_{z_n} into two pieces $Z_{[0,\sigma_n]}\mu_{z_n}, Z_{[\sigma_n,1]}\mu_{z_n}$ and concatenating one of the pieces onto the other branch.
Secondly, after a sequence of renormalizations, the branches of a distortion measure pair look like concatenations of zooms of the original distortion measures. The δ-boundedness conditions guarantees that after the sign of the sequence $\underline{z}$ changes three times, there comes a point where the pieces have all been acted upon by zooms with subinterval sizes uniformly bounded in terms of δ.
Thirdly, using the Zoom Contraction Law and a uniform bound on the norm of renormalizations, the distance between the renormalized distortion measure pairs on each piece can be shown to contract by a uniform $\kappa < 1$.
Finally, as $(\mu_-, \mu_+), (\nu_-, \nu_+)$ share the same canonical projection, the renormalizations have the same measure on every piece, so the net contribution of any piece to the distance is zero. Hence, the overall distance contracts by κ.

Our main goal is to apply this contraction theorem in the setting of Möbius circle maps. The main theorem is the following:

Convergence Theorem. *Let f be a Möbius circle map in the attractor for renormalization, of total distortion K. Then f has a unique $\mathcal{R}$-invariant limiting distortion measure pair (μ_-, μ_+) associated to it.*

To do this, we consider sequences of preimages of Möbius circle maps for which the weights remain uniformly bounded. If the weights of f have the same sign, this can be done in a unique way. We define the set $W_{K,\delta}$ to be those sequences for which the total distortion equals K, and the renormalization scheme for the sequence is δ-bounded. It is possible to show that every point in the attractor for renormalization satisfies these conditions. We then consider the space $\Gamma_{K,\delta}$ of bounded graphs from $W_{K,\delta}$ into $\mathcal{M} \times \mathcal{M}$ that respect the canonical projection. Using the completeness of $\mathcal{M} \times \mathcal{M}$ and the renormalization schemes associated to every point in $W_{K,\delta}$, which are all δ-bounded by assumption, one defines a graph transform operator on $\Gamma_{K,\delta}$ and shows that it is a contraction on a complete space, hence it has a unique fixed point. Extending this result over the union of all $W_{K,\delta}$ provides renormalization invariant distortion measure pairs for any Möbius circle map with a combinatorially bounded history.

Now that one has constructed invariant distortion measure pairs, it is natural to try and understand their properties. This thesis presents two partial and relatively straightforward results in that direction. First of all, if $K = 0$, then the invariant measures are identically zero, and if $K \neq 0$, it is possible to show that the distortion measure pairs associated to a sequence $\underline{f} \in W_{K,\delta}$ are either both strictly positive or both strictly negative, the choice of which agrees with the sign of K. Additionally, if the renormalization scheme of $\underline{f}$ satisfies a stronger condition satisfied by Möbius circle maps with bounded combinatorics, then its invariant distortion measure pairs are *dyadic doubling*, i.e., the measures on any pair of standard dyadic intervals of level n that share a dyadic ancestor at level $n-1$ are proportional by a uniform constant which is independent of n.

2 Renormalization of Möbius Circle Maps

2.1 Renormalization of Circle Maps

2.1.1 Definition of Circle Maps

Definition 1. *A **circle map** is a branched interval map of $[a,b] \subseteq \mathbb{R}$ to itself, parametrized by two marked points $c, v \in (a,b)$ and two orientation preserving homeomorphisms $f_- : [a,c] \to [v,b]$ and $f_+ : [c,b] \to [a,v]$. We refer to $[a,b]$ as the **domain of definition**, and f_-, f_+ as the **branches** of a circle map $f = (a,b,c,v,f_-,f_+)$.*

Consider an orientation preserving homeomorphism f of $\mathbb{S}^1$ with one marked point x_0. We can then choose any interval $[a,b]$ and take a lift F of f onto $[a,b]$ by identifying x_0 with the endpoints of the interval. Then, one can define c and v as the images of $f^{-1}(x_0)$ and $f(x_0)$, respectively, under the circle identification. Note that F is a branched interval map on $[a,b]$ with a point of discontinuity at c. By definition, F is a homeomorphism when restricted to $[a,c)$ and $[c,b)$, so we can define $f_- : [a,c] \to [v,b]$, $f_+ : [c,b] \to [a,v]$ that agree with F on their common domain of definition and extend naturally to the closure via $f_-(c) = b, f_+(b) = v$. From this process, we can embed f as a circle map (a,b,c,v,f_-,f_+), which justifies our definition.

If we place stronger conditions on the branches f_-, f_+, for instance, that they are diffeomorphisms, then a circle map f as we have defined them can only correspond to maps of $\mathbb{S}^1$ with at most two nonsmooth points.

2.1.2 Classical Renormalization of Circle Maps

Definition 2. *Let $f : [a,b] \to [a,b]$ be an interval map. If $[c,d] \subseteq [a,b]$, then we define the **first return map** of f to $[c,d]$ to be the map $f_{\text{fr}} : [c,d] \to [c,d]$ such that $f_{\text{fr}}(x) = f^n(x)$, where n is the minimal integer such that $f^n(x) \in [c,d]$, if it exists.*

Definition 3. *Let $f = (a,b,c,v,f_-,f_+)$ be a circle map. If $c \neq v$, we say that f is **renormalizable**. We define the **prerenormalization** of f as the circle map $p\mathcal{R}f$ corresponding to the first return map of f to the interval $[v,b]$, if $c > v$, or $[a,v]$, if $c < v$.*

Lemma 1.

$$p\mathcal{R}f = \begin{cases} (v,b,c,f_-(v),f_-,f_- \circ f_+) & \text{if } c > v, \\ (a,v,c,f_+(v),f_+ \circ f_-,f_+) & \text{if } v > c. \end{cases}$$

Proof. We will prove the statement for the case $c > v$. The case $c < v$ will follow by a similar argument. So suppose $c > v$. Then by definition, prerenormalization acts as the first return map to the interval $[v, 1]$, which gives us the domain of definition. The point of discontinuity remains inside $[v, 1]$, so its position does not change. As a result, the $-$ branch is defined on $[v, c]$ and the $+$ branch on $[c, 1]$. The image of $(f_-)_{|[v,c]}$ is $[f_-(v), 1]$, so we observe that the point corresponding to v in $p\mathcal{R}f$ will be $f_-(v)$. Finally, f_+ maps the interval $[c, 1]$ outside the domain of definition, so to obtain a first return map, we further iterate by f_-. That makes the prerenormalized $+$ branch equal to $f_- \circ f_+$, which completes the argument that $p\mathcal{R}f = (v, 1, c, f_-(v), f_-, f_- \circ f_+)$. □

At this point, we should probably introduce the convention that $[a, b] = [0, 1]$ and that renormalization is the method to preserve this structure. Then we can refer to p without reference to a and b.

Definition 4. *Let $f = (c, v, f_-, f_+)$ be a circle map. If f is renormalizable, we define the **renormalization** of f to be the circle map $\mathcal{R}f$ whose coordinates are obtained by applying the affine orientation preserving conjugation to the coordinates of $p\mathcal{R}f$ that maps its domain of definition onto $[0, 1]$.*

Remark. *$\mathcal{R}f$ is a circle map.*

It is probably best to refrain from defining $\mathcal{R}$ as an operator right now, since we haven't yet defined the space of circle maps as a Banach space.

2.2 Nonlinearities and Renormalization

2.2.1 Introduction

Definition 5. *If $f : [a, b] \to [c, d]$ is an orientation preserving homeomorphism, we define the **normalization** of f to be the orientation preserving homeomorphism $Nf : [0, 1] \to [0, 1]$ given by the formula*

$$Nf(x) = \frac{f(a + (b - a)x) - c}{d - c}.$$

Remark. *An orientation preserving homeomorphism f is determined uniquely by its domain, range and normalization. Thus, since the domain and ranges of f_-, f_+ are predefined, we can use Nf_-, Nf_+ instead as coordinates of f.*

Proposed convention: $F_\pm := Nf_\pm$ when used inside the coordinates of a circle map f.

2.2.2 Definition of the Nonlinearity

Definition 6. *Let $f : [a, b] \to [c, d]$ be a C^2 orientation preserving diffeomorphism. The **nonlinearity** of f is the C^0 function $\eta_f : [a, b] \to \mathbb{R}$ given by*

$$\eta_f(x) = \frac{f''(x)}{f'(x)}.$$

We denote by $\mathrm{Diff}^k([0, 1])$ the Banach space of orientation preserving C^k diffeomorphisms of $[0, 1]$, $k \geq 2$, with norm given by $||F|| = ||\eta_F||_0$. For $\eta \in C^0([0, 1])$, we define $F_\eta \in \mathrm{Diff}^2([0, 1])$ to be the unique orientation preserving diffeomorphism of $[0, 1]$ such that $\eta_{F_\eta} = \eta$.

It is important to remark that the vector space structure of $\mathrm{Diff}^2([0,1])$ corresponds to addition of nonlinearities rather than compositions of diffeomorphisms.

Definition 7. *Let $F \in \mathrm{Diff}^2([0,1])$. The **weight** of F is the quantity*

$$\omega_F := \int_0^1 \eta_F.$$

2.2.3 The Chain Rule of Nonlinearity

Lemma 2 (Chain Rule of Nonlinearity)**.** *Let $F, G \in \mathrm{Diff}^2([0,1])$. Then*

$$\eta_{F\circ G}(x) = \eta_F(G(x))G'(x) + \eta_G(x).$$

Proof. Suppose that $F, G \in \mathrm{Diff}^2([0,1])$. Then by the Chain Rule, $(F \circ G)'(x) = F'(G(X))G'(x)$. Further applying the Chain Rule, we observe that $(F \circ G)''(x) = F''(G(x))[G'(x)]^2 + F'(G(x))G''(x)$. Hence,

$$\eta_{F\circ G}(x) = \frac{(F \circ G)''(x)}{(F \circ G)'(x)} = \frac{F''(G(x))}{F'(G(x))}G'(x) + \frac{G''(x)}{G'(x)} = \eta_F(G(x))G'(x) + \eta_G(x).$$

□

Proposition 1. *Let $L > 0$. If $F, G \in \mathrm{Diff}^2([0,1])$ satisfy $||\eta_F||, ||\eta_G|| < L$, then there exists a constant C depending on L such that $|F - G|, |F' - G'| < C||\eta_F - \eta_G||$.*

Proof. First, observe that by the Chain Rule of Nonlinearities, $\eta_F(x) - \eta_G(x) = \eta_{F\circ G^{-1}}(G(x))G'(x)$, so it will suffice to write $|F - G|$, $|F' - G'|$ in terms of $\eta_{F\circ G^{-1}}$. Since $F(x) = F(G^{-1}(G(x)))$,

$$|(F - G)(x)| = |(F \circ G^{-1} - \mathrm{Id})(G(x))| \leq |F \circ G^{-1} - \mathrm{Id}| < |(F \circ G^{-1} - \mathrm{Id})'|.$$

As

$$(F \circ G^{-1})'(G(x)) = (F \circ G^{-1})'(0)e^{\int_0^{G(x)} \eta_{F\circ G^{-1}}} = (F \circ G^{-1})'(0)e^{\int_0^x \eta_F - \eta_G},$$

we obtain that $|F - G|$ is bounded by a constant times $||\eta_F - \eta_G||$.
Similarly, $|(F' - G')(x)| = G'(x)|(F \circ G^{-1})'(G(x)) - 1|$, which is also bounded by a constant times $||\eta_F - \eta_G||$, with the constant depending on L. This completes the proof. □

2.2.4 Definition of the Zoom Operator

Definition 8. *Let $I \subseteq [0,1]$. We define the **zoom operator** $Z_I : \mathrm{Diff}^2([0,1]) \to \mathrm{Diff}^2([0,1])$ as the normalization of the restriction operation $F|_I : I \to F(I)$, i.e.,*

$$Z_I F = N(F|_I).$$

Lemma 3. *If $I = [a,b] \subseteq [0,1]$ and $F \in \mathrm{Diff}^2([0,1])$, then*

$$\eta_{Z_I F}(x) = |I|\eta_F(a + (b - a)x).$$

In particular, $||Z_I F|| \leq |I|||F||$.

Proof. Suppose that $I = [a, b] \subseteq [0, 1]$ and $F \in \text{Diff}^2([0, 1])$. Then

$$Z_I F(x) = N(F_{|I})(x) = \frac{F(a + (b - a)x) - F(a)}{F(b) - F(a)},$$

so $(Z_I F)'(x) = \dfrac{(b - a)F'(a + (b - a)x)}{F(b) - F(a)}$ and
$(Z_I F)''(x) = \dfrac{(b - a)^2 F''(a + (b - a)x)}{F(b) - F(a)}$. Therefore,

$$\eta_{Z_I F}(x) = (b - a)\eta_F(a + (b - a)x) = |I|\eta(a + (b - a)x).$$

From this, it follows that $||Z_I F|| \leq |I|||\eta_F||_0 = |I|||F||$. □

Remark. *For any $I = [a, b] \subseteq [0, 1]$, $Z_I : C^0([0, 1]) \to C^0([0, 1])$,*

$$Z_I \eta(x) = |I|\eta(a + (b - a)x)$$

defines an operator on the space of C^0 nonlinearities, with $\int_0^1 Z_I \eta = \int_I \eta$.

2.2.5 Extension of Renormalization

Definition 9. *Let $f = (c, v, F_-, F_+)$ be a circle map. We refer to (c, v) as the **spatial coordinates** of f and the C^0 functions (η_{F_-}, η_{F_+}) as the **nonlinearity coordinates** of f. By identification of f with its spatial and nonlinearity coordinates, we let $X = (0, 1)^2 \times C^0([0, 1])^2$ denote the space of circle maps, endowed with the product distance.*

By convention, we will refer to $\eta_\pm$ as the nonlinearity of $F_\pm$.

Definition 10. *Let $f = (c, v, \eta_-, \eta_+) \in X$. We define $X_- := \{f \in X : c > v\}$ and $X_+ := \{f \in X : c < v\}$. If $c \neq v$, we say that f is **renormalizable**, and for such f, we define the **renormalization operator** $\mathcal{R} : X_- \sqcup X_+ \to X$ as follows:*

$$\mathcal{R}f = \begin{cases} (\frac{c - v}{1 - v}, F_{\eta_-}(\frac{v}{c}), Z_{[v/c,1]}\eta_-, \eta_{Z_{[0,v/c]}F_- \circ F_+}) & \text{if } c > v, \\ (\frac{c}{v}, F_{\eta_+}(\frac{v - c}{1 - c}), \eta_{Z_{[(v-c)/(1-c),1]}F_+ \circ F_-}, Z_{[0,(v-c)/(1-c)]}\eta_+) & \text{if } c < v. \end{cases}$$

2.3 Möbius Diffeomorphisms and their Properties

2.3.1 Definition of Möbius Diffeomorphisms

Definition 11. *We define the **Möbius diffeomorphism** of weight $\omega \in \mathbb{R}$ to be the linear fractional transformation $M_\omega \in \text{Diff}^2([0, 1])$ given by*

$$M_\omega(x) = \frac{x}{(1 - e^{\omega/2})x + e^{\omega/2}}.$$

Remark. *The Möbius diffeomorphisms form the union of all orientation preserving linear fractional transformations that map $[0, 1]$ onto itself.*

Lemma 4. 1) *The nonlinearity* $\eta_\omega \in C^0([0,1])$ *of* M_ω *is given by the formula*

$$\eta_\omega(x) = \frac{2(e^{\omega/2}-1)}{(1-e^{\omega/2})x + e^{\omega/2}}.$$

2) $\int_0^1 \eta_\omega = \omega$.

Proof. 1) Observe that $M'_\omega(x) = \dfrac{e^{\omega/2}}{((1-e^{\omega/2})x+e^{\omega/2})^2}$, and $M''_\omega(x) = \dfrac{-2e^{\omega/2}(1-e^{\omega/2})}{((1-e^{\omega/2})x+e^{\omega/2})^3}$, making

$$\eta_\omega(x) = \frac{2(e^{\omega/2}-1)}{(1-e^{\omega/2})x + e^{\omega/2}}.$$

2) By direct integration of η_ω,

$$\int_0^1 \eta_\omega = [-2\ln((1-e^{\omega/2})x + e^{\omega/2})]_0^1 = 0 - 2\frac{\omega}{2} = \omega.$$

□

2.3.2 General Properties of Möbius Diffeomorphisms

Lemma 5. *Let* $\omega_1, \omega_2 \in \mathbb{R}$, *and let* $[a,b] \subseteq [0,1]$. *Then:* 1)

$$M_{\omega_1+\omega_2} = M_{\omega_1} \circ M_{\omega_2}.$$

2)

$$\int_a^b \eta_{\omega_1} - \eta_{\omega_2} = \int_{M_{\omega_2}(a)}^{M_{\omega_2}(b)} \eta_{\omega_1-\omega_2}.$$

Proof. 1) Note first that the product of two Möbius diffeomorphisms is itself a Möbius diffeomorphism, since they all are orientation preserving linear fractional transformations that fix 0 and 1. It remains to determine the weight of $M_{\omega_1} \circ M_{\omega_2}$, which by the Chain Rule of Nonlinearity and a change of variables equals

$$\int_0^1 \eta_{\omega_1}(M_{\omega_2})M'_{\omega_2} + \eta_{\omega_2} = \int_0^1 \eta_{\omega_1} + \int_0^1 \eta_{\omega_2} = \omega_1 + \omega_2.$$

2) Let $[a,b] \subseteq [0,1]$. Then by part 1), the Chain Rule of Nonlinearity and a change of variables,

$$\int_a^b \eta_{\omega_1} - \eta_{\omega_2} = \int_a^b \eta_{M_{\omega_1-\omega_2}\circ M_{\omega_2}} - \eta_{\omega_2} = \int_a^b \eta_{\omega_1-\omega_2}(M_{\omega_2})M'_{\omega_2}$$
$$= \int_{M_{\omega_2}(a)}^{M_{\omega_2}(b)} \eta_{\omega_1-\omega_2}.$$

□

2.3.3 Useful Formulas

Lemma 6.

$$M_\omega(x) = \frac{x}{e^{(\int_x^1 \eta_\omega)/2}} = \frac{e^{(\int_0^x \eta_\omega)/2} - 1}{e^{\omega/2} - 1}.$$

Proof. Observe that by integration, $\int_x^1 \eta_\omega = 2\ln((1-e^{\omega/2})x + e^{\omega/2})$, so

$$e^{\int_x^1 \eta_\omega/2} = (1-e^{\omega/2})x + e^{\omega/2} = \frac{x}{M_\omega(x)},$$

which proves the first equality. For the second equality, note that
$e^{\int_0^x \eta_\omega/2} = \dfrac{e^{\omega/2}}{e^{\int_x^1 \eta_\omega/2}}$, so

$$\begin{aligned} e^{\int_0^x \eta_\omega/2} - 1 &= \frac{e^{\omega/2} - ((1-e^{\omega/2})x + e^{\omega/2})}{(1-e^{\omega/2})x + e^{\omega/2}} = \frac{x(e^{\omega/2}-1)}{(1-e^{\omega/2})x + e^{\omega/2}} \\ &= (e^{\omega/2}-1)M_\omega(x), \end{aligned}$$

which after dividing by $e^{\omega/2} - 1$ completes the proof. □

Remark. $D\eta_\omega(x) = \dfrac{\eta_\omega^2(x)}{2}$*, as linear fractional transformations are characterized by having Schwarzian derivative* 0*, and the Schwarzian derivative associated to a* C^1 *nonlinearity* η *is given by the operator*
$S : C^1([0,1]) \to C^0([0,1]), S\eta(x) = D\eta(x) - \eta^2(x)/2.$

2.4 Hyperbolic Theory for Möbius Circle Maps

2.4.1 Definition and Invariance of the Total Distortion

Definition 12. *Let* $f = (c, v, \eta_-, \eta_+) \in X$*. We define the* ***total distortion*** *of* f *to be the quantity* $K(f) := \int_0^1 \eta_- + \int_0^1 \eta_+$*.*

Proposition 2. *Suppose that* $f \in X$ *is renormalizable. Then* $K(\mathcal{R}f) = K(f)$*.*

Proof. We will prove this when $c > v$. We calculate:

$$\begin{aligned} K(\mathcal{R}f) &= \int_0^1 \eta_-(\mathcal{R}f) + \int_0^1 \eta_+(\mathcal{R}f) = \int_0^1 Z_{[v/c,1]}\eta_- + \int_0^1 \eta Z_{[0,v/c]}F_- \circ F_+ \\ &= \int_{v/c}^1 \eta_- + \int_0^{v/c} \eta_- + \int_0^1 \eta_+ = \int_0^1 \eta_- + \int_0^1 \eta_+ = K(f). \end{aligned}$$

The proof for $c < v$ is similar. □

2.4.2 The Hyperbolicity Theorem

Definition 13. *Let* $K \in \mathbb{R}$*. We let* X_K *denote the space of* $f \in X$ *such that* $K(f) = K$*. We further define* $\Omega_K \subset X_K$ *to be the subset of circle maps with Möbius nonlinearities in* X_K*. Finally, we let* $\Omega = \sqcup_K \Omega_K$*.*

Remark. *The Möbius nonlinearities are not a subspace of* $C^0([0,1])$ *since they are not closed under addition. However, by the bijection* $(\eta_{\omega_-}, \eta_{\omega_+}) \mapsto (\omega_-, \omega_+) \in \mathbb{R}^2$*, they can be endowed with a Banach space structure.*

Piece 1. *Let X_K^1 consist of those circle maps in X_K with C^1 nonlinearities. Then as $n \to \infty$, $\mathcal{R}^n(X_K^1) \to \Omega_K$.*

Piece 2. 1) *As $n \to \infty, \mathcal{R}^n(\Omega_0)$ converges to the space*

$$R_\rho := \{(\rho, 1-\rho, Id, Id) : \rho \in (0,1)\}$$

of rigid rotations.
2) *If $K \neq 0$, then $\mathcal{R}$ is a hyperbolic operator on Ω_K. The unstable manifolds of $\mathcal{R}$ form a lamination homeomorphic to the product of $\mathbb{R}$ with a Cantor set. The unstable manifolds are dimension 1 and consist of all points $p \in \Omega_K$ which have a uniformly bounded sequence of preimages.*

3 Distortion Measures and their Renormalization

3.1 Introduction to Distortion

3.1.1 Weight Structure of Distortion

In this section, we restrict ourselves to Ω_K, and develop the bijection into $\mathbb{R}^2$ of the weight pairs.

Lemma 7. *Let $f = (c, v, \omega_-, \omega_+) \in \Omega_K$. Then,*

$$(\omega_-(\mathcal{R}f), \omega_+(\mathcal{R}f)) = \begin{cases} (\int_{v/c}^1 \eta_{\omega_-}, \omega_+ + \int_0^{v/c} \eta_{\omega_-}) & \text{if } c > v, \\ (\omega_- + \int_{\frac{v-c}{1-c}}^1 \eta_{\omega_+}, \int_0^{\frac{v-c}{1-c}} \eta_{\omega_+}) & \text{if } c < v. \end{cases}$$

Proof. The proof follows from the fact that $\int_0^1 Z_I \eta_\omega = \int_I \eta_\omega$. □

3.1.2 Scaling Ratios and Related Coordinates

Definition 14. *Let $c, v \in (0,1)$. We define the **scaling ratios** of c and v as $\sigma_- = \dfrac{v}{c}, \sigma_+ = \dfrac{v-c}{1-c}$.*

Remark.

$$c = \frac{\sigma_+}{\sigma_- + \sigma_+ - 1}, v = \frac{\sigma_-\sigma_+}{\sigma_- + \sigma_+ - 1}.$$

Lemma 8. *Let $f = (c, v, \eta_-, \eta_+) \in X_K$. Then*

$$(\sigma_-(\mathcal{R}f), \sigma_+(\mathcal{R}f)) = \begin{cases} (\frac{F_-(\sigma_-)(1-\sigma_+)}{-\sigma_+}, \sigma_+ + (1-\sigma_+)F_-(\sigma_-)) & \text{if } \sigma_- < 1, \\ (F_+(\sigma_+)\sigma_-, \frac{\sigma_- F_+(\sigma_+) - 1}{\sigma_- - 1}) & \text{if } \sigma_- > 1. \end{cases}$$

Proof. As auxiliary calculations, note that $1 - v = \dfrac{(1-\sigma_-)(\sigma_+ - 1)}{\sigma_- + \sigma_+ - 1}$ and $1 - c = \dfrac{\sigma_- - 1}{\sigma_- + \sigma_+ - 1}$ Now we split the proof into two cases. First, assume that $\sigma_- < 1$, and note that this implies $c > v$. Then

$$\sigma_-(\mathcal{R}f) = \frac{v(\mathcal{R}f)}{c(\mathcal{R}f)} = \frac{F_-(\sigma_-)}{(c-v)/(1-v)} = \frac{F_-(\sigma_-)(1-\sigma_-)(\sigma_+-1)}{\sigma_+(1-\sigma_-)}$$
$$= F_-(\sigma_-)\frac{\sigma_+-1}{\sigma_+},$$

$$\sigma_+(\mathcal{R}f) = \frac{v(\mathcal{R}f)-c(\mathcal{R}f)}{1-c(\mathcal{R}f)} = \frac{F_-(\sigma_-)(1-v)-(c-v)}{1-c}$$
$$= \frac{(\sigma_- -1)(\sigma_+ + (1-\sigma_+)F_-(\sigma_-))}{\sigma_- -1} = \sigma_+ + F_-(\sigma_-)(1-\sigma_+).$$

Next, assume that $\sigma_- > 1$ and note that this implies $c < v$. Then

$$\sigma_-(\mathcal{R}f) = \frac{v(\mathcal{R}f)}{c(\mathcal{R}f)} = F_+(\sigma_+)\frac{v}{c} = F_+(\sigma_+)\frac{\sigma_-\sigma_+}{\sigma_+} = \sigma_- F_+(\sigma_+),$$

$$\sigma_+(\mathcal{R}f) = \frac{v(\mathcal{R}f)-c(\mathcal{R}f)}{1-c(\mathcal{R}f)} = \frac{vF_+(\sigma_+)-c}{v-c} = \frac{\sigma_+(\sigma_-F_+(\sigma_+)-1)}{\sigma_+(\sigma_- -1)}$$
$$= \frac{\sigma_-F_+(\sigma_+)-1}{\sigma_- -1}.$$

□

3.1.3 Injectivity of Renormalization Branches

Lemma 9. *Let $f \in \Omega_K$. Then there exist at most two $f_-, f_+ \in \Omega_K$ such that $\sigma_-(f_-) < 1, \sigma_-(f_+) > 1$ and $\mathcal{R}f_- = f = \mathcal{R}f_+$.*

Proof. Let $f = (c, v, \omega_-, \omega_+) \in \Omega_K$ and let σ_-, σ_+ be its associated scaling ratios. We will first prove uniqueness of f_-. To prove that $\mathcal{R}$ is injective on that branch, we assume there exist $c_-, v_- \in (0,1), \omega_{1,-} \in \mathbb{R}$ such that $c_- > v_-$, $c = (c_- - v_-)/(1-v_-), v = M_{\omega_{1,-}}(v_-/c_-)$ and $\omega_- = \int_{\sigma_-}^1 \eta_{\omega_{1,-}}$. Let
$\sigma_{1,-} = v_-/c_-$. By Lemma 6,

$$v = \frac{\sigma_{1,-}}{e^{\omega_-/2}},$$

so $\sigma_{1,-}$ is fixed, and therefore, so is $\omega_{1,-}$. Note that by assumption, $\sigma_{1,-} \in (0,1)$. Thus, this implies $v_- = \sigma_{1,-}c_-$, so we obtain that $c = \dfrac{c_-(1-\sigma_{1,-})}{1-\sigma_{1,-}c_-}$, which gives

$$c_- = \frac{c}{1-(1-c)\sigma_{1,-}} \in (0,1),$$

also fixing v_-. Finally, we select $\omega_{1,+}$ such that $\omega_{1,+} + \omega_{1,-} = K$, since the total distortion is preserved by renormalization. Hence, we have found a unique $f_- = (c_-, v_-, \omega_{1,-}, \omega_{1,+}) \in \Omega_K$ such that $\mathcal{R}f_- = f$ and
$\sigma_-(f_-) < 1$.
Next, we want to prove uniqueness of f_+. So, we first suppose there exist $c_+, v_+ \in (0,1), \omega_{2,+} \in \mathbb{R}$ such that $c_+ < v_+, c = \dfrac{c_+}{v_+}$, and for $\sigma_{2,+} = \dfrac{v_+ - c_+}{1-c_+}$, we

additionally obtain that $v = M_{\omega_{2,+}}(\sigma_{2,+}), \omega_+ = \int_0^{\sigma_{2,+}} \eta_{\omega_{2,+}}$. As a consequence of Lemma 6,

$$v = \frac{e^{\omega_+/2} - 1}{e^{\omega_{2,+}/2} - 1},$$

which fixes $\omega_{2,+}$, and therefore also $\sigma_{2,+}$. By assumption, $\omega_{2,+}$ is assumed to exist, so $e^{\omega_{2,+}/2} - 1 > -1$. Solving the equations

$$c = \frac{c_+}{v_+}, \frac{v_+ - c_+}{1 - c_+} = \sigma_{2,+}$$

simultaneously for c_+ and v_+ gives

$$v_+ = \frac{\sigma_{2,+}}{1 - (1 - \sigma_{2,+})c} \in (0, 1), c_+ = cv_+.$$

Finally, we let $f_+ = (c_+, v_+, \omega_{2,-}, \omega_{2,+}) \in \Omega_K$ by setting $\omega_{2,-} + \omega_{2,+} = K$. By the preservation of the total distortion, $\mathcal{R}f_+ = f$, and since f_+ is uniquely defined, the proof is complete. □

Note that I restrict myself to the Möbius case for this argument, since my proof depends on the precise formulas for Möbius diffeomorphisms.

Lemma 10. *Let* $f = (c, v, \omega_-, \omega_+) \in \Omega_K$. *Then*
1) *There exists* $f_- \in \Omega_K$ *such that* $\mathcal{R}f_- = f$ *and* $\sigma_-(f_-) < 1$ *if and only if* $ve^{\omega_-/2} < 1$.
2) *There exists* $f_+ \in \Omega_K$ *such that* $\mathcal{R}f_+ = f$ *and* $\sigma_-(f_+) > 1$ *if and only if* $e^{\omega_+/2} > 1 - v$.

Proof. 1) Assume that $f_- = (c_-, v_-, \omega_{1,-}, \omega_{1,+})$ has scaling ratio $\sigma_{1,-}$ and satisfies $\mathcal{R}f_- = f$. Then, by the proof of the previous lemma, if $c_- > v_-$, it follows that $\sigma_{1,-} = ve^{\omega_-/2}$. Thus, $\sigma_{1,-} \in (0, 1)$ if and only if $ve^{\omega_-/2} < 1$, which was what we wanted.
2) Assume that $f_+ = (c, +, v_+, \omega_{2,-}, \omega_{2,+})$ has scaling ratios $\sigma_{2,-}, \sigma_{2,+}$. By the proof of the previous lemma, if $\sigma_{2,-} > 1$, it follows that $\mathcal{R}f_+ = f$ if and only if $-1 < e^{\omega_{2,+}/2} - 1 = \dfrac{e^{\omega_+/2} - 1}{v}$, i.e., $e^{\omega_+/2} > 1 - v$. This completes the proof. □

Remark. *If* ω_- *and* ω_+ *have the same sign, then any* $f \in \Omega$ *with weights* (ω_-, ω_+) *has a preimage in* Ω. *The same holds true if* $\omega_- \leq 0,\ \omega_+ \geq 0$.

Note that $\Omega_-,\ \Omega_+$ are symmetric to one another, which can be discerned by using the map

$$c \mapsto 1 - c, v \mapsto 1 - v, \omega_- \mapsto -\omega_+, \omega_+ \mapsto -\omega_-.$$

Under this mapping, it can been seen that the conditions for surjectivity are symmetric, as one would expect. The symmetry will also allow us to simplify proofs, since a result for renormalization in one component will work for the other with minor modification.

3.2 Temporal Structure of Distortion

3.2.1 Introduction to Decompositions

Definition 15. *Let $\mathcal{T}$ denote the set of dyadic rationals in $(0,1]$. A* ***nonlinearity decomposition*** *is a function $\underline{\eta} = (\eta_\tau)_{\tau\in\mathcal{T}}$ into the space of C^0 nonlinearities such that $\sum_{\tau\in\mathcal{T}} ||\eta_\tau||_0 < \infty$. For every $n \in \mathbb{N}$, define*

$$F_n(\underline{\eta}) = F_1 \circ F_{(2^n-1)/2^n} \circ \ldots \circ F_{2^{-n}} \in \mathrm{Diff}^2([0,1]),$$

where $F_\tau = F_{\eta_\tau}$, and let $F(\underline{\eta}) = \lim_{n\to\infty} F_n(\underline{\eta}) \in \mathrm{Diff}^2([0,1])$, if the limit exists.

Definition 16. *If $\underline{\eta_1}, \underline{\eta_2}$ are nonlinearity decompositions, then we define $\underline{\eta_1} \star \underline{\eta_2} = (\eta_\tau)_{\tau\in\mathcal{T}}$ to be the nonlinearity decomposition given by the formula*

$$\eta_\tau = \begin{cases} \eta_{1,2\tau} & \text{if } \tau \leq 1/2, \\ \eta_{2,2\tau-1} & \text{if } \tau > 1/2. \end{cases}$$

Remark. *For all $n \in \mathbb{N}$,*

$$F_n(\underline{\eta_1} \star \underline{\eta_2}) = F_{n-1}(\underline{\eta_2}) \circ F_{n-1}(\underline{\eta_1}).$$

3.2.2 Decomposition of Weights

Definition 17. *A* ***weight decomposition*** *is a function $\underline{\omega} = (\omega_\tau)_{\tau\in\mathcal{T}}$ into the space of real numbers such that $\sum_{\tau\in\mathcal{T}} |\omega_\tau| < \infty$.*

Remark. *If $\underline{\omega}$ is a weight decomposition, we associate to it the nonlinearity decomposition $\underline{\eta}$ such that $\eta_\tau = \eta_{\omega_\tau}$ is Möbius.*

Definition 18. *For any $\tau_0 \in \mathcal{T}$ and any weight decomposition $\underline{\omega}$, we define $\underline{\omega_{(0,\tau_0]}}$ to be the restriction $\underline{\tilde{\omega}}$ such that*

$$\tilde{\omega}_\tau = \begin{cases} \omega_\tau & \text{if } \tau \leq \tau_0, \\ 0 & \text{if } \tau > \tau_0. \end{cases}$$

Definition 19. *Let $\mathcal{A}$ denote the σ-algebra generated by the half open intervals $(\tau_1, \tau_2]$ with endpoints in $\{0\} \cup \mathcal{T}$. For any weight decomposition $\underline{\omega}$, we define $\mu_{\underline{\omega}}$ as the signed atomic Borel measure on $\mathcal{A}$ such that for all $\tau \in \mathcal{T}$,*

$$\mu_{\underline{\omega}}(\{\tau\}) = \omega_\tau.$$

Remark. $\mu_{\underline{\omega}}(0,\tau_0] = \sum_{\tau\leq\tau_0} \omega_\tau$.

Remark. *For any $I \subseteq [0,1]$ and any nonlinearity decomposition $\underline{\eta}$, we can define the zoom decomposition $Z_I\underline{\eta}$ in a manner that preserves the property*

$$F(Z_I\underline{\eta}) = Z_I F(\underline{\eta}).$$

Pointwise, $(Z_I\eta)_{\tau_0} = Z_{I(\tau_0)}\eta_{\tau_0}$, where $I(\tau_0)$ depends on $|I|$ and $\sum_{\tau<\tau_0} ||\eta_\tau||_0$. Using zoom decompositions and concatenations, one can define renormalization on a lift of X where the nonlinearity coordinates η_+, η_- are replaced by nonlinearity decompositions $\underline{\eta_-}$, $\underline{\eta_+}$ that satisfy

$$F_\pm = F(\underline{\eta_\pm}).$$

In an analogous manner, we can define renormalization on a lift of Ω where the weights ω_- and ω_+ are replaced by weight decompositions $\underline{\omega_-}$, $\underline{\omega_+}$ such that

$$\sum_{\tau \in \mathcal{T}} \omega_{\pm,\tau} = \omega_{\pm}.$$

Because of the additivity of weights under composition, it turns out that it is natural in this setting to analyze the measures $\mu_{\underline{\omega_\pm}}$, since for any $\tau \in \mathcal{T}$ and any $I \subseteq [0,1]$,

$$(\mu_{Z_I \underline{\omega}})(0,\tau] = \int_I \eta_{\mu_{\underline{\omega}}(0,\tau]}.$$

Since renormalization acts by zooming and concatenating, and zooming contracts $\sup_{\tau \in \mathcal{T}} |\omega_\tau|$ it can be shown that in the lift of Ω, the measures $\mu_{\underline{\omega_\pm}}$ converge under renormalization to non-atomic measures. This motivates us to try and extend the definitions of renormalization to this larger measure space.

3.3 Distortion Measures

3.3.1 Definition of Distortion and Spacetime Measures

Definition 20. *A* ***distortion measure*** *is a signed Borel measure on $\mathcal{A}$, equipped with the* ***distortion norm***

$$||\mu|| = \sup_{\tau \in \mathcal{T}} |\mu(0,\tau]|.$$

Let $\mathcal{M} = \{\mu : ||\mu|| < \infty\}$ denote the space of distortion measures, equipped with the distortion norm.

Remark. *The distortion norm is different from the usual total variation norm for signed measures, since its magnitude is only taken over the generating intervals for the sigma algebra $\mathcal{A}$. However, it still defines a norm, since the scaling property and triangle inequality are easily seen to be satisfied, while $||\mu|| = 0$ implies that $\mu(0,\tau] = 0$ for all $\tau \in \mathcal{T}$, from which it follows that the same property holds for all intervals $(\tau_1, \tau_2]$ with endpoints in $\mathcal{T}$, and thus $\mu(A) = 0$ on any countable union or countable intersection A of such sets.*

A ***spacetime measure*** *is a pair $(\mu, (\eta_\tau)_{\tau \in \mathcal{T}})$ composed of a distortion measure μ and a function $(\eta_\tau)_{\tau \in \mathcal{T}}$ into the space of L^1 nonlinearities such that*

$$\int_0^1 \eta_\tau = \mu(0,\tau], \sup_{\tau \in \mathcal{T}} \int_0^1 |\eta_\tau| < \infty.$$

We refer to $||(\mu, (\eta_\tau)_{\tau \in \mathcal{T}})|| = \sup_{\tau \in \mathcal{T}} \int_0^1 |\eta_\tau|$ as the ***spacetime norm****. We will denote the space of spacetime measures endowed with the spacetime norm by $\underline{\mathcal{M}}$.*

Remark. *The nonlinearities which make up a spacetime measure do not form a nonlinearity decomposition, since for example $\sum_{\tau \in \mathcal{T}} ||\eta_\tau||_0$ could be infinite. However, when it makes sense, for a nonlinearity decomposition $\underline{\eta_0}$, one could define a spacetime measure $(\mu, (\eta_\tau)_{\tau \in \mathcal{T}})$ such that $\eta_\tau = \eta_{F(\underline{\eta_{0,(0,\tau]}})}$, where $\underline{\eta_{(0,\tau]}}$ is defined analogously to the weight decomposition case. In this case,*

$$\eta_1 = \eta_{F(\underline{\eta_0})}.$$

Remark. *If $(\mu, (\eta_\tau)_{\tau \in \mathcal{T}}) \in \underline{\mathcal{M}}$, then*

$$||(\mu, (\eta_\tau)_{\tau \in \mathcal{T}})|| \geq ||\mu||.$$

3.3.2 Operations on Spacetime Measures

Definition 21. *Let $(\mu, (\eta_\tau)_{\tau\in\mathcal{T}}) \in \underline{\mathcal{M}}$. For any $I \subseteq [0,1]$, we define the* ***zoom operator*** $Z_I : \underline{\mathcal{M}} \to \underline{\mathcal{M}}$ *by the definition*

$$(Z_I(\mu, (\eta_\tau)_{\tau\in\mathcal{T}})) = (Z_I\mu, (Z_I\eta_\tau)_{\tau\in\mathcal{T}}),$$

where Z_I denotes the zoom operator on nonlinearities, and

$$(Z_I\mu)(0,\tau] = \int_I \eta_\tau.$$

Definition 22. *Let $(\mu_1, (\eta_{1,\tau})_{\tau\in\mathcal{T}}), (\mu_2, (\eta_{2,\tau})_{\tau\in\mathcal{T}})$ be spacetime measures. We define the* ***concatenation operator*** $\star : \underline{\mathcal{M}} \times \underline{\mathcal{M}} \to \underline{\mathcal{M}}$ *as follows:*

$$(\mu_1, (\eta_{1,\tau})_{\tau\in\mathcal{T}}) \star (\mu_2, (\eta_{2,\tau})_{\tau\in\mathcal{T}}) = (\mu_1 \star \mu_2, ((\eta_1 \star \eta_2)_\tau)_{\tau\in\mathcal{T}}),$$

where

$$(\eta_1 \star \eta_2)_\tau = \begin{cases} \eta_{1,2\tau} & \text{if } \tau \leq 1/2, \\ \eta_{F_{\eta_{2,2\tau-1}} \circ F_{\eta_{1,1}}} & \text{if } \tau > 1/2, \end{cases}$$

$$(\mu_1 \star \mu_2)(0,\tau] = \begin{cases} \mu_1(0, 2\tau] & \text{if } \tau \leq 1/2, \\ \mu_1(0,1] + \mu_2(0, 2\tau - 1] & \text{if } \tau > 1/2. \end{cases}$$

3.4 Renormalization of Möbius Measure Pairs

3.4.1 Definition of Möbius Measures

Definition 23. *The spacetime measure $(\mu, (\eta_\tau)_{\tau\in\mathcal{T}})$ is* ***Möbius*** *if for every $\tau \in \mathcal{T}$, η_τ is a Möbius nonlinearity.*

Proposition 3 (Möbius Identification)**.** *If $(\mu, (\eta_\tau)_{\tau\in\mathcal{T}})$ is Möbius, then*

$$||(\mu, (\eta_\tau)_{\tau\in\mathcal{T}})|| = ||\mu||,$$

and we can identify $\mathcal{M}$ with the space of Möbius measures.

Proof. Suppose that $(\mu, (\eta_\tau)_{\tau\in\mathcal{T}})$ is Möbius. As Möbius nonlinearities are either strictly positive or strictly negative, $\int_0^1 |\eta_\tau| = |\int_0^1 \eta_\tau| = |\mu(0,\tau]|$. Thus,

$$||(\mu, (\eta_\tau)_{\tau\in\mathcal{T}})|| = \sup_{\tau\in\mathcal{T}} \int_0^1 |\eta_\tau| = \sup_{\tau\in\mathcal{T}} |\mu(0,\tau]| = ||\mu||.$$

It follows that we can embed $\mathcal{M}$ into the space of spacetime measures via the identification $\mu \mapsto (\mu, (\eta_{\mu(0,\tau]})_{\tau\in\mathcal{T}})$. □

Lemma 11. *$Z_I : \mathcal{M} \to \mathcal{M}$ and $\star : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ extend as operators via the Möbius Identification, with*

$$(Z_I\mu)(0,\tau] = \int_I \eta_{\mu(0,\tau]}, (\mu_1 \star \mu_2)(0,\tau] = \begin{cases} \mu_1(0, 2\tau] & \text{if } \tau \leq 1/2, \\ \mu_1(0,1] + \mu_2(0, 2\tau - 1] & \text{if } \tau > 1/2. \end{cases}$$

Proof. This is a direct consequence of the definition of Z_I and $\star$ on $\underline{\mathcal{M}}$. □

3.4.2 The Space of Distortion Measure Pairs

Definition 24. *Let $\mathcal{M} \times \mathcal{M}$ denote the space of distortion measure pairs, endowed with the norm $||(\mu_-, \mu_+)|| = \max\{||\mu_-||, ||\mu_+||\}$.*

Lemma 12. *$\mathcal{M} \times \mathcal{M}$ is a Banach space.*

Proof. It suffices to prove completeness of $\mathcal{M}\times\mathcal{M}$. Let $(\mu_{n,-}, \mu_{n,+})$ be a Cauchy sequence of distortion measure pairs. Then for any $\tau \in \mathcal{T}$, $(\mu_{n,-}(0, \tau]), (\mu_{n,+}(0, \tau])$ are Cauchy sequences of real numbers, so they converge in $\mathbb{R}$. We then define the measures μ_-, μ_+ on $\mathcal{A}$ via the definition

$$\mu_-(0, \tau] = \lim_{n\to\infty} \mu_{n,-}(0, \tau], \mu_+(0, \tau] = \lim_{n\to\infty} \mu_{n,+}(0, \tau].$$

Then for any $\epsilon > 0$, there exists $n \in \mathbb{N}$ sufficiently large so that for all $\tau \in \mathcal{T}$, $|(\mu_- - \mu_{n,-})(0, \tau]|, |(\mu_+ - \mu_{n,+})(0, \tau]| < \epsilon$, so $||(\mu_-, \mu_+)||$ is uniformly bounded. Therefore, $(\mu_-, \mu_+) \in \mathcal{M} \times \mathcal{M}$, which completes the proof of completeness. □

3.4.3 The Renormalization Operators for Distortion Measure Pairs

Now we have constructed a space $\mathcal{M} \times \mathcal{M}$ of distortion measure pairs, and we understand how to act on it, via zooming and concatenation. What remains is to define renormalization in a way that is consistent with its definition on Ω. The first step is to define a projection operator that will allow us to build the lift of Ω.

Definition 25. *Let $\pi : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^2$ denote the* ***canonical projection***

$$\pi(\mu_-, \mu_+) = (\mu_-(0, 1], \mu_+(0, 1]).$$

Next we need to check that the canonical projection commutes with the zoom and concatenation operators on $\mathcal{M} \times \mathcal{M}$, but that follows directly from their definition. As a result, we can look at the formulas for $(\omega_-(\mathcal{R}f), \omega_+(\mathcal{R}f))$ and use them as a blueprint. Note that the choice of renormalization for $f \in \Omega$ depends only on the sign of $\ln(\sigma_-)$ and one scaling ratio, so we will have to define two separate operators which both depend on $\sigma \in (0, 1)$. Recalling that

$$(\omega_-(\mathcal{R}f), \omega_+(\mathcal{R}f)) = \begin{cases} (\int_{\sigma_-}^1 \eta_{\omega_-}, \omega_+ + \int_0^{\sigma_-} \eta_{\omega_-}) & \text{if } \sigma_- < 1, \\ (\omega_- + \int_{\sigma_+}^1 \eta_{\omega_+}, \int_0^{\sigma_+} \eta_{\omega_+}) & \text{if } \sigma_- > 1, \end{cases}$$

we define:

Definition 26. *Let $\sigma \in (0, 1)$ and let $(\mu_-, \mu_+) \in \mathcal{M} \times \mathcal{M}$. We define the* ***renormalization operators*** *$\mathcal{R}^\sigma_\pm : \mathcal{M} \times \mathcal{M} \circlearrowleft$ by*

$$\mathcal{R}^\sigma_-(\mu_-, \mu_+) = (Z_{[\sigma,1]}\mu_-, \mu_+ \star Z_{[0,\sigma]}\mu_-),$$
$$\mathcal{R}^\sigma_+(\mu_-, \mu_+) = (\mu_- \star Z_{[\sigma,1]}\mu_+, Z_{[0,\sigma]}\mu_+).$$

4 Contraction of Distortion Measure Fibers

4.1 Fiber Structure of Distortion Measures

Definition 27. *For any $p = (p_-, p_+) \in \mathbb{R}^2$ we define the **slice** $(\mathcal{M} \times \mathcal{M})_p \in \mathcal{M} \times \mathcal{M}$ to be the set of distortion measure pairs (μ_-, μ_+) such that $\pi(\mu_-, \mu_+) = p$. For $K \in \mathbb{R}$, we define*

$$(\mathcal{M} \times \mathcal{M})_K = \{(\mathcal{M} \times \mathcal{M})_p : p_- + p_+ = K\}.$$

Remark. *The slices $(\mathcal{M} \times \mathcal{M})_p$, $p \in \mathbb{R}^2$ form a fibration of $\mathcal{M} \times \mathcal{M}$. Any renormalization operator $\mathcal{R}_z^\sigma$ preserves this fibration, as well as $(\mathcal{M} \times \mathcal{M})_K$.*

Definition 28. *Let $f \in \Omega$. We define $z(f) = \operatorname{sign}(\ln(\sigma_-(f)))$. We let $s : \Omega \to (0, 1) \times \{+, -\}$ denote the mapping*

$$s(f) = (\sigma_{z(f)}(f), z(f)).$$

Remark. *Let $Y = (0, 1) \times \{-, +\} \times \mathcal{M} \times \mathcal{M}$. Then we have a commutative diagram*

$$\begin{array}{ccc} {}_{s}\,Y & \xrightarrow{\mathcal{R}_z^\sigma} & Y \\ {\scriptstyle \pi^{-1}}\uparrow & & \downarrow{\scriptstyle \pi} \\ \Omega & \xrightarrow{\mathcal{R}} & \Omega \end{array}$$

4.2 Distortion of Renormalized Möbius Measure Fibers

4.2.1 The Zoom Contraction Law

Proposition 4 (Zoom Contraction Law)**.** *Let $L > 0$ and suppose that $\mu, \nu \in \mathcal{M}$ satisfy $||\mu||, ||\nu|| < L$. Then for any I compactly contained in $[0, 1]$, there exists $\kappa < 1$ depending only on L and $|I|$ such that*

$$||Z_I\mu - Z_I\nu|| < \kappa||\mu - \nu||.$$

Proof. Assume that $L > 0$ and that $\mu, \nu \in \mathcal{M}$ satisfy $||\mu||, ||\nu|| < L$. Let $I \subseteq [0, 1]$, $I \neq [0, 1]$. For any $\tau \in \mathcal{T}$, note that by part 2) of Lemma 5,

$$(Z_I\mu - Z_I\nu)(0, \tau] = \int_I \eta_{\mu(0,\tau]} - \eta_{\nu(0,\tau]} = \int_{M_{\nu(0,\tau]}(I)} \eta_{(\mu-\nu)(0,\tau]},$$

where $M_{\nu(0,\tau]}(I)$ has uniformly bounded size depending on $|I|$ and L. Now observe that for any $x, y \in (0, 1)$ and any $\omega \in \mathbb{R}, |\ln(\eta_\omega(x)/\eta_\omega(y))| \leq |\omega|/2$. Since $|(\mu - \nu)(0, \tau]| < 2L$, it follows that

$$\frac{\int_{M_{\nu(0,\tau]}(I)} \eta_{(\mu-\nu)(0,\tau]}}{\int_0^1 \eta_{(\mu-\nu)(0,\tau]}} < \frac{e^L |M_{\nu(0,\tau]}(I)|}{e^L |M_{\nu(0,\tau]}(I)| + (1 - |M_{\nu(0,\tau]}(I)|)} := \kappa,$$

independent of τ. Thus, there exists $\kappa < 1$ depending only on L, $|I|$ such that

$$||Z_I\mu - Z_I\nu|| < \kappa||\mu - \nu||.$$

□

4.2.2 The Concatenation Lemma

Lemma 13 (Concatenation Lemma)**.** *Let $\mu, \nu \in \mathcal{M}$. Then*

$$||\mu \star \nu|| \leq \max\{||\mu||, |\mu(0,1]| + ||\nu||\}.$$

Proof. Assume that $\mu, \nu \in \mathcal{M}$. Then by definition, for any $\tau \in \mathcal{T}$,

$$(\mu \star \nu)(0,\tau] = \begin{cases} \mu(0,2\tau] & \text{if } \tau \leq 1/2, \\ \mu(0,1] + \nu(0,2\tau-1] & \text{if } \tau > 1/2. \end{cases}.$$

In either case, it is apparent that $|(\mu \star \nu)(0,\tau]| \leq \max\{||\mu||, |\mu(0,1]| + ||\nu||\}$, which after taking a supremum over $\tau \in \mathcal{T}$ completes the proof. □

4.2.3 Basic Contraction Principle

Proposition 5. *Let $L > 0$ and suppose that $(\mu_-, \mu_+), (\nu_-, \nu_+) \in \mathcal{M} \times \mathcal{M}$ satisfy $||(\mu_-, \mu_+)||, ||(\nu_-, \nu_+)|| < L, (\mu_-, \mu_+), (\nu_-, \nu_+) \in (\mathcal{M} \times \mathcal{M})_p$ for $p \in \mathbb{R}^2$. Then, for any renormalization operator $\mathcal{R}_z^\sigma$,*

$$||\mathcal{R}_z^\sigma(\mu_-, \mu_+) - \mathcal{R}_z^\sigma(\nu_-, \nu_+)|| \leq ||(\mu_-, \mu_+) - (\nu_-, \nu_+)||,$$

and additionally, if z_1, z_2 have opposite sign, and $\delta \in (0, 1/2)$ exists such that $\sigma_1, \sigma_2 \in (\delta, 1-\delta)$, then there exists $\kappa < 1$ depending only on L and δ such that

$$||(\mathcal{R}_{z_2}^{\sigma_2} \circ \mathcal{R}_{z_1}^{\sigma_1})(\mu_-, \mu_+) - (\mathcal{R}_{z_2}^{\sigma_2} \circ \mathcal{R}_{z_1}^{\sigma_1})(\nu_-, \nu_+)|| < \kappa||(\mu_-, \mu_+) - (\nu_-, \nu_+)||.$$

Proof. Let $L > 0, (\mu_-, \mu_+), (\nu_-, \nu_+) \in \mathcal{M} \times \mathcal{M}$ satisfy

$$||(\mu_-, \mu_+)||, ||(\nu_-, \nu_+)|| < L, \pi(\mu_-, \mu_+) = \pi(\nu_-, \nu_+).$$

Let $\mathcal{R}_z^\sigma$ be a renormalization operator. Without loss of generality, suppose that $z = -$. We will consider both branches of $\mathcal{R}_-^\sigma(\mu_-, \mu_+) - \mathcal{R}_-^\sigma(\nu_-, \nu_+)$ separately. On the $-$ branch, $\mathcal{R}_-^\sigma$ acts by $Z_{[\sigma,1]}$, so by the Zoom Contraction Law, there exists κ_- depending only on $1-\sigma$ and L such that the distance between the $-$ branches of $\mathcal{R}_-^\sigma(\mu_-, \mu_+)$ and $\mathcal{R}_-^\sigma(\nu_-, \nu_+)$ is contracted by a factor of κ_-. For the $+$ branch, we can exploit that

$$(\mu_+ \star Z_{[0,\sigma]}\mu_-) - (\nu_+ \star Z_{[0,\sigma]}\nu_-) = (\mu_+ - \nu_+) \star (Z_{[0,\sigma]}\mu_- - Z_{[0,\sigma]}\nu_-),$$

so as $\pi(\mu_-, \mu_+) = \pi(\nu_-, \nu_+), (\mu_+ - \nu_+)(0,1] = 0$, it follows from the Concatenation Lemma and the Zoom Contraction Law that there exists $\kappa_+ < 1$ depending only on L and σ such that
$||(\mu_+ - \nu_+) \star (Z_{[0,\sigma]}\mu_- - Z_{[0,\sigma]}\nu_-)|| \leq \max\{||\mu_+ - \nu_+||, \kappa_+||\mu_- - \nu_-||\}$. By setting $\kappa = \max\{\kappa_-, \kappa_+\}$, we obtain that

$$||\mathcal{R}_-^\sigma(\mu_-, \mu_+) - \mathcal{R}_-^\sigma(\nu_-, \nu_+)|| \leq \max\{||\mu_+ - \nu_+||, \kappa||\mu_- - \nu_-||\},$$

where κ depends on σ and L. By symmetry,

$$||\mathcal{R}_+^\sigma(\mu_-, \mu_+) - \mathcal{R}_+^\sigma(\nu_-, \nu_+)|| \leq \max\{||\mu_- - \nu_-||, \kappa||\mu_+ - \nu_+||\}$$

for the same $\kappa < 1$. This completes the proof of the first statement. Now suppose that that $\delta \in (0, 1/2)$ exists such that $\sigma_1, \sigma_2 \in (\delta, 1-\delta)$. Note that

$||\mathcal{R}_-^{\sigma_1}(\mu_-,\mu_+)||, ||\mathcal{R}_-^{\sigma_1}(\nu_-,\nu_+)|| < 2L$ and $\pi(\mathcal{R}_-^{\sigma_1}(\mu_-,\mu_+)) = \pi(\mathcal{R}_-^{\sigma_1}(\nu_-,\nu_+))$. Hence, by taking a larger κ if necessary to account for the $2L$ discrepancy, we apply the proof of the previous statement twice to show that there exists $\kappa < 1$ depending only on δ and L such that

$$||(\mathcal{R}_+^{\sigma_2} \circ \mathcal{R}_-^{\sigma_1})(\mu_-,\mu_+) - (\mathcal{R}_+^{\sigma_2} \circ \mathcal{R}_-^{\sigma_1})(\nu_-,\nu_+)|| \leq$$
$$\max\{||Z_{[\sigma_1,1]}\mu_- - Z_{[\sigma_1,1]}\nu_-||, \kappa||(\mu_+ - \nu_+) \star (Z_{[0,\sigma_1]}\mu_- - Z_{[0,\sigma_1]}\nu_-)||\}$$
$$\leq \kappa||(\mu_-,\mu_+) - (\nu_-,\nu_+)||.$$

By symmetry, the same inequality holds for $\mathcal{R}_-^{\sigma_2} \circ \mathcal{R}_+^{\sigma_1}$, completing the proof. □

4.3 Asymptotics of Renormalization Schemes

4.3.1 Definition of Renormalization Schemes

Definition 29. *Let* $(\underline{\sigma}, \underline{z}) \in (0,1)^{\mathbb{N}} \times \{+,-\}^{\mathbb{N}}$. *Such sequence pairs shall be referred to as* ***renormalization schemes***. *For any* $n \in \mathbb{N}$, *we define the operator* $\mathcal{R}^n(\underline{\sigma}, \underline{z}) : \mathcal{M} \times \mathcal{M} \circlearrowleft$ *to be*

$$\mathcal{R}^n(\underline{\sigma}, \underline{z}) = \mathcal{R}_{z_1}^{\sigma_1} \circ ... \circ \mathcal{R}_{z_n}^{\sigma_n}.$$

4.3.2 The Fast Renormalization Subsequence

Definition 30. *For any renormalization scheme* $(\underline{\sigma}, \underline{z})$ *we define the* ***fast renormalization subsequence*** *of* $(\underline{\sigma}, \underline{z})$ *to be the maximal subsequence* (σ_{n_k}, z_{n_k}) *of* $(\underline{\sigma}, \underline{z})$ *such that* $z_{n_k} \neq z_{n_k+1}$. *For every* $k \in \mathbb{N}$, *we define the operator* $\mathcal{R}_{\text{fast}}^k(\underline{\sigma}, \underline{z}) = \mathcal{R}^{n_k}(\underline{\sigma}, \underline{z})$ *to be the k-th* ***fast renormalization operator*** *associated to* $(\underline{\sigma}, \underline{z})$.

4.3.3 δ-bounded Combinatorics and their Meaning

Definition 31. *For any* $\delta \in (0, 1/2)$, *we say that a renormalization scheme* $(\underline{\sigma}, \underline{z})$ *is* δ*-bounded if the following three conditions are satisfied:*
1) *For the fast renormalization subsequence* (σ_{n_k}, z_{n_k}),

$$z_{n_k} = - \Rightarrow \sigma_{n_k+1} < 1 - \delta,$$
$$z_{n_k} = + \Rightarrow \sigma_{n_k+1} > \delta.$$

2)

$$z_n = - \text{ and } \sigma_n > 1 - \delta \Rightarrow \sigma_{n-1} > 1 - \delta \text{ or } z_{n-1} = +,$$
$$z_n = + \text{ and } \sigma_n < \delta \Rightarrow \sigma_{n-1} < \delta \text{ or } z_{n-1} = -.$$

3) *For every* $i, j \in \mathbb{N}$ *such that* $\sigma_i < \delta$, $\sigma_j > 1 - \delta$, *there exists* $k \in \mathbb{N}$ *between* i *and* j *such that* $\sigma_k \in (\delta, 1 - \delta)$.

4.3.4 Contraction of Fast Renormalization for δ-bounded Schemes

Lemma 14. *Let* $L > 0$, *and let* $(\underline{\sigma}, \underline{z})$ *be a renormalization scheme. Then if* $||(\mu_-,\mu_+)|| < L$, *it follows that* $||\mathcal{R}^n(\underline{\sigma}, \underline{z})(\mu_-,\mu_+)|| < 2L$ *for every* $n \in \mathbb{N}$.

Proof. Let $L > 0$, and let $\mathcal{R}_z^\sigma$ be any signed renormalization operator. Let $(\mu_-, \mu_+) \in \mathcal{M} \times \mathcal{M}$ and suppose that $||(\mu_-, \mu_+)|| < L$. We define a new norm $||\cdot||_S$ such that

$$||(\mu_-, \mu_+)||_S = \max\{||\mu_-||_- + ||\mu_+||_-, ||\mu_-||_+ + ||\mu_+||_+\},$$

where $||\mu||_- = \max\{0, -\inf_{\tau \in \mathcal{T}} \mu(0, \tau]\}, ||\mu||_+ = \max\{0, \sup_{\tau \in \mathcal{T}} \mu(0, \tau]\}$.
First, observe that $||(\mu_-, \mu_+)|| \leq ||(\mu_-, \mu_+)||_S$, since $||\mu|| = \max\{||\mu||_-, ||\mu||_+\}$. Now, we claim that $||\mathcal{R}_z^\sigma(\mu_-, \mu_+)||_S \leq ||(\mu_-, \mu_+)||_S$. We start by observing that for either choice z of sign and any $I \subseteq [0,1]$, $||Z_I\mu||_z = |\int_I \eta_{||\mu||_z}|$. Thus, for any $\sigma \in (0,1), ||Z_{[0,\sigma]}\mu||_z + ||Z_{[\sigma,1]}\mu||_z = ||\mu||_z$. Additionally,
$||\mu \star \nu||_z \leq ||\mu||_z + ||\nu||_z$. As all signed renormalization operators consist of concatenations and complementary zooms, it follows that $\mathcal{R}_z^\sigma$ cannot expand $||\cdot||_S$. Since $||(\mu_-, \mu_+)||_S < 2L$, and this quantity is not expanded by $\mathcal{R}^n(\underline{\sigma}, \underline{z})$, we obtain that

$$||\mathcal{R}^n(\underline{\sigma}, \underline{z})(\mu_-, \mu_+)|| \leq ||\mathcal{R}^n(\underline{\sigma}, \underline{z})(\mu_-, \mu_+)||_S \leq ||(\mu_-, \mu_+)||_S < 2L,$$

which was what we wanted. □

Theorem 1 (Contraction Theorem)**.** *Let $(\underline{\sigma}, \underline{z})$ be a δ-bounded sequence. Then for any $L > 0$, there exists $\kappa < 1$ depending only on δ and L such that for every $(\mu_-, \mu_+), (\nu_-, \nu_+) \in \mathcal{M} \times \mathcal{M}$ that satisfies*

$$||(\mu_-, \mu_+)||, ||(\nu_-, \nu_+)|| < L, (\mu_-, \mu_+), (\nu_-, \nu_+) \in (\mathcal{M} \times \mathcal{M})_p$$

for $p \in \mathbb{R}^2$,

$$||\mathcal{R}^3_{\text{fast}}(\mu_-, \mu_+) - \mathcal{R}^3_{\text{fast}}(\nu_-, \nu_+)|| < \kappa||(\mu_-, \mu_+) - (\nu_-, \nu_+)||.$$

Proof. Suppose that $(\underline{\sigma}, \underline{z})$ is δ-bounded and let $L > 0$. Now let $(\mu_-, \mu_+), (\nu_-, \nu_+)$ be distortion measure pairs such that $||(\mu_-, \mu_+)||, (\nu_-, \nu_+)|| < L$ and
$\pi(\mu_-, \mu_+) = \pi(\nu_-, \nu_+) = p \in \mathbb{R}^2$. Let (σ_{n_k}, z_{n_k}) be the fast renormalization subsequence of $(\underline{\sigma}, \underline{z})$. Without loss of generality, assume that $z_{n_1} = -, z_{n_2} = +, z_{n_3} = -$. For $k \in [1,3]$, let $(\mu_{k,-}, \mu_{k_+}), (\nu_{k,-}, \nu_{k,+})$ denote the distortion measure pairs attained by applying the operators composing $\mathcal{R}^3_{\text{fast}}(\underline{\sigma}, \underline{z})$ in their correct order, i.e.,

$$(\mu_{3,-}, \mu_{3,+}) = \mathcal{R}^3_{\text{fast}}(\underline{\sigma}, \underline{z})(\mu_-, \mu_+) = \mathcal{R}^k_{\text{fast}}(\underline{\sigma}, \underline{z})(\mu_{k,-}, \mu_{k,+}),$$
$$(\nu_{3,-}, \nu_{3,+}) = \mathcal{R}^3_{\text{fast}}(\underline{\sigma}, \underline{z})(\nu_-, \nu_+) = \mathcal{R}^k_{\text{fast}}(\underline{\sigma}, \underline{z})(\nu_{k,-}, \nu_{k,+}).$$

Our goal is to prove that there exists $\kappa < 1$ depending only on L and δ such that $||(\mu_{3,-} - \nu_{3,-}, \mu_{3,+} - \nu_{3,+})|| < \kappa||(\mu_- - \nu_-, \mu_+ - \nu_+)||$. By Lemma 14, we know that $||(\mu_{k,-}, \mu_{k,+})||, ||(\nu_{k,-}, \nu_{k,+})|| < 2L$ for all $k \in [1,3]$. We begin by estimating $||(\mu_{2,-} - \nu_{2,-}, \mu_{2,+} - \nu_{2,+})||$. Consider the scaling ratios $\sigma_{n_2+1}, ..., \sigma_{n_3}$, for which the corresponding z are all negative by assumption. From the definition of δ-boundedness, $\sigma_{n_2+1} \geq \delta$, since $z_{n_2} = +$ and none of $\sigma_{n_2+2}, ..., \sigma_{n_3}$ lie in $(1-\delta, 1)$. Now we split the proof into two cases, depending on whether $\sigma_{n_2+1} > 1 - \delta$.

First, suppose that $\sigma_{n_2+1} > 1-\delta$. By repeated application of the first statement of the Basic Contraction Principle and the Zoom Contraction Law, there exists $\kappa_2 < 1$ depending only on $1-\delta$ such that $||\mu_{2,-} - \nu_{2,-}|| < \kappa_2||\mu_- - \nu_-||$

and $||\mu_{2,+} - \nu_{2,+}|| < \max\{||\mu_+ - \nu_+||, ||\mu_- - \nu_-||\}$. Now consider the scaling ratios $\sigma_{n_1+1}, ..., \sigma_{n_2}$, for which the corresponding z are all positive. Since $\sigma_{n_2+1} > 1-\delta$, by the definition of δ-boundedness, there exists $k \in [n_1+1, n_2]$ such that $\sigma_k \in (\delta, 1-\delta)$ and $\sigma_i > \delta$ for $i \in [k, n_2]$. Now we apply the renormalization operator $\mathcal{R}_+^{\sigma_k} \circ ... \circ \mathcal{R}_+^{\sigma_{n_2}}$ to $(\mu_{2,-}, \mu_{2,+}), (\nu_{2,-}, \nu_{2,+})$. Observe that this operator acts as a composition of zooms on the $+$ branch and applies concatenations to the $-$ branch by pieces of the old $+$ branch acted upon by the zoom operators $Z_{[\sigma_i,1]}$ for $i \in [k, n_2]$. We claim there exists $\kappa_1 < 1$ depending only on δ and L such that

$$||(\mathcal{R}_+^{\sigma_k} \circ ... \circ \mathcal{R}_+^{\sigma_{n_2}})(\mu_{2,-}, \mu_{2,+}) - (\mathcal{R}_+^{\sigma_k} \circ ... \circ \mathcal{R}_+^{\sigma_{n_2}})(\nu_{2,-}, \nu_{2,+})|| < \max\{\kappa_1 ||\mu_{2,+} - \nu_{2,+}||, ||\mu_{2,-} - \nu_{2,-}||\}.$$

The claim holds true for the $+$ branch because $\sigma_k \in (\delta, 1-\delta)$ while as the pieces concatenated to the $-$ branch are acted upon by zoom operators with zoom interval sizes bounded by $1-\delta$, they also contract by a definite factor depending only on δ and L. Moreover, since renormalization operators maps slices to slices, we can utilize that the distortion norm of a measure composed of concatenations of measures whose mass over $(0, 1]$ is 0 is the max of the distortion norms of its pieces. This completes the proof of the claim. Now, we apply the first statement of the Basic Contraction Principle repeatedly to $\mathcal{R}_+^{\sigma_{n_1+1}} \circ ... \circ \mathcal{R}_+^{\sigma_{i-1}}$ to obtain via nonexpansion of the distortion norm that

$$\begin{aligned} ||(\mu_{1,-} - \nu_{1,-}, \mu_{1,+} - \nu_{1,+})|| &< \max\{||\mu_{2,-} - \nu_{2,-}||, \kappa_1 ||\mu_{2,+} - \nu_{2,+}||\} \\ &< \max\{\kappa_1, \kappa_2\} ||(\mu_- - \nu_-, \mu_+ - \nu_+)||. \end{aligned}$$

Since $\mathcal{R}_{\text{fast}}(\underline{\sigma}, \underline{z})$ does not expand the distortion norm, it follows by taking $\kappa = \max\{\kappa_1, \kappa_2\}$ that

$$||(\mu_{3,-} - \nu_{3,-}, \mu_{3,+} - \nu_{3,+})|| \leq ||(\mu_{1,-} - \nu_{1,-}, \mu_{1,+} - \nu_{1,+})|| < \kappa ||(\mu_- - \nu_-, \mu_+ - \nu_+)||,$$

which was what we wanted.

Note that the argument actually shows that if $z_{n_3} = -$ and $\sigma_{n_2+1} > 1-\delta$, then we observe contraction in two fast steps. By symmetry, the same holds for the same $\kappa < 1$ if $z_{n_3} = +$ and $\sigma_{n_2+1} < \delta$. Now we assume $\sigma_{n_2+1} \in (\delta, 1-\delta)$. Again, we can choose $\kappa_2 < 1$ depending only on δ and L so that

$$||\mu_{2,-} - \nu_{2,-}|| < \kappa_2 ||\mu_- - \nu_-||, ||\mu_{2,+} - \nu_{2,+}|| \leq ||(\mu_- - \nu_-, \mu_+ - \nu_+)||.$$

Now consider the scaling ratios $\sigma_{n_1+1}, ..., \sigma_{n_2}$. If there exists $k \in [n_1+1, n_2]$ such that $\sigma_k \in (\delta, 1-\delta)$, we apply the proof of the previous case to show that $||(\mu_{1,-} - \nu_{1,-}, \mu_{1,+} - \nu_{1,+})|| < \kappa ||(\mu_- - \nu_-, \mu_+ - \nu_+)||$ for the same $\kappa < 1$ as before. Otherwise, $\sigma_{n_1+1} < \delta$, which by symmetry implies that $\mathcal{R}^2_{\text{fast}}(\underline{\sigma}, \underline{z})$ acts as a contraction, since our argument could then be shown to work in two fast steps. Hence,

$$||\mathcal{R}^3_{\text{fast}}(\mu_-, \mu_+) - \mathcal{R}^3_{\text{fast}}(\nu_-, \nu_+)|| < \kappa ||(\mu_-, \mu_+) - (\nu_-, \nu_+)||.$$

By symmetry, the same argument holds if $z_{n_3} = +$ and $\sigma_{n_2+1} \in (\delta, 1-\delta)$, which completes the proof. □

5 Distortion Measures for Unstable Manifolds

5.1 Introduction

5.2 Bounded Renormalization Schemes

5.2.1 Definition of Backwards Sequences and their Associated Schemes

Definition 32. *Let $f \in \Omega_K$, and suppose that $\omega_-(f), \omega_+(f)$ both have the same sign as K. For each $\underline{z} \in \{+,-\}^{\mathbb{N}}$, we define the sequence $\underline{f}_{\underline{z}}$ to be the unique backwards sequence $(f_n)_{n\in\mathbb{N}}$ of preimages of f in Ω_K such that* $\text{sign}(\ln(\sigma_-(f_n))) = z_n$*, and $\mathcal{R}f_n = f_{n-1}$, if it exists. We refer to $(\sigma_{z_n}(f_n), z_n)_{n\in\mathbb{N}}$ as the* ***renormalization scheme*** *of $\underline{f}_{\underline{z}}$.*

Proposition 6. *Let $f = (c, v, \omega_-, \omega_+) \in \Omega_K$ for $K \neq 0$. If ω_-, ω_+ have the same sign, then there exists a unique word z for which the weights of $\underline{f}_{\underline{z}}$ all have the same sign.*

Proof. Assume without loss of generality that $K > 0$ and suppose that the weights of $f = (c, v, \omega_-, \omega_+) \in \Omega_K$ are both positive. It will suffice to show that f has a unique inverse for which the weights are bounded in magnitude by K. First, suppose that f has no inverse in Ω_-, i.e., $ve^{\omega_-/2} > 1$. Let $\tilde{\omega_+}$ denote the $+$ weight of the preimage of f in Ω_+. By Lemma 6,

$$e^{\omega_+/2} - 1 \geq e^{-\omega_-/2}(e^{\tilde{\omega_+}/2} - 1),$$

hence $\tilde{\omega_+} - \omega_- \leq \omega_+$, i.e., $\tilde{\omega_+} < K$, which proves that the weights of the inverse of f in Ω_+ are both positive.
Next, suppose that $\tilde{\omega_+} > K$. This implies by our previous statement that f has an inverse in Ω_-. Let $\tilde{\sigma_-}$ denote the scaling ratio for this inverse. Since $v = \dfrac{\tilde{\sigma_-}}{e^{\omega_-/2}}$ by Lemma 6, we obtain that

$$\begin{aligned} \frac{e^{\omega_+/2} - 1}{v} &> e^{K/2} - 1 \\ \Leftrightarrow e^{K/2} - e^{\omega_-/2} &> \tilde{\sigma_-}(e^{K/2} - 1) \\ \Leftrightarrow e^{K/2}(1 - \tilde{\sigma_-}) &> e^{\omega_-/2} - \tilde{\sigma_-} \\ \Leftrightarrow e^{K/2}(1 - \tilde{\sigma_-}) + \tilde{\sigma_-} &> e^{\omega_-/2}. \end{aligned}$$

But if $\tilde{\omega_-}$ denotes the $-$ weight of the preimage of f in Ω_-, then we know from Lemma 6 that $e^{\omega_-/2} = e^{\tilde{\omega_-}/2}(1 - \tilde{\sigma_-}) + \tilde{\sigma_-}$,so it follows that

$$\tilde{\omega_+} > K \Leftrightarrow \tilde{\omega_-} < K.$$

By flipping the inequality, we obtain in a similar manner that

$$\tilde{\omega_-} > K \Leftrightarrow \tilde{\omega_+} < K.$$

This completes the proof for $K > 0$, while if $K < 0$, the argument follows from the previous case by Möbius symmetry. □

I plan on omitting the reference to $\underline{z}$ when the combinatorics are not important, and will simply refer to backwards sequences as $\underline{p}$ for brevity.

Remark. *If $f \in \Omega_K$ has a renormalization scheme, then it extends to $\mathcal{M} \times \mathcal{M}$ for the same $(\underline{\sigma}, \underline{z})$.*

5.2.2 Hierarchy of Bounded Sequence Spaces and their Properties

Definition 33. *For each $K \in \mathbb{R}$ and each $\delta \in (0, 1/2)$, we define $W_{K,\delta}$ to be the set of all backwards sequences $\underline{f}_{\underline{z}} \in \Omega_K^{\mathbb{N}}$ such that firstly, the weights $\omega_-(f_n), \omega_+(f_n)$ remain uniformly bounded for all $n \in \mathbb{N}$, and secondly, the renormalization scheme of $\underline{f}_{\underline{z}}$ is δ-bounded. We then define $W_K = \cup_{\delta \in (0,1/2)} W_{K,\delta}$, and $W = \sqcup_{K \in \mathbb{R}} W_K$.*

Proposition 7. *Let $K \neq 0$. For each $f \in \Omega_K$ contained in the attractor of renormalization, there exists a backwards sequence $\underline{f}_{\underline{z}} \in W_K$.*

Proof. Let $K \neq 0$, and suppose without loss of generality that $K > 0$. Suppose that $f \in \Omega_K$ lies in the attractor of renormalization. Then the weights $\omega_-(f), \omega_+(f)$ are both positive, so by Proposition 6 there exists a unique sequence $\underline{f}$ of preimages of f whose weights are all uniformly bounded. Let $(\underline{\sigma}, \underline{z})$ be the renormalization scheme associated to $\underline{f}$. Using a well known fact from one-dimensional dynamics, we claim that the fast renormalizations f_{n_k} associated to the fast renormalization subsequence have bounded distortion. We will now clarify what that entails.

Recall that the branches M_-, M_+ of a Möbius circle map f are normalized maps. The original maps are linear fractional transformations f_-, f_+ whose domains and ranges depend on the spatial coordinates of f. For any C^1 diffeomorphism $g : I \to g(I)$, one can define its **slope** $s(g) = \dfrac{|g(I)|}{|I|}$, and with the slope, one obtains the formula

$$g'(y) = s(g) N g'(\frac{y - a}{b - a}),$$

where $I = [a, b]$. The slopes of f_-, f_+ are given by the formulas

$$s_- = \frac{1 - v}{c} = \frac{(1 - \sigma_+)(\sigma_- - 1)}{\sigma_+}, \; s_+ = \frac{v}{1 - c} = \frac{\sigma_- \sigma_+}{\sigma_- - 1}.$$

As a consequence of bounded distortion, there exists a constant $C > 1$ such that the derivatives $f'_{n_k,-}, f'_{n_k,+}$ are bounded between $1/C$ and C. As the weights $\omega_\pm(f_{n_k})$ are uniformly bounded by K, this entails that the slopes $s_-(f_{n_k}), s_+(f_{n_k})$ are uniformly bounded.

We will now present the formulas for the renormalization of slopes:

$$(\mathcal{R}s_-, \mathcal{R}s_+) = \begin{cases} (s_- \dfrac{1 - M_{\omega_-}(\sigma_-)}{1 - \sigma_-}, s_+ s_- \dfrac{M_{\omega_-}(\sigma_-)}{\sigma_-}) & \text{if } \sigma_- < 1, \\ (s_- s_+ \dfrac{1 - M_{\omega_+}(\sigma_+)}{1 - \sigma_+}, s_+ \dfrac{M_{\omega_+}(\sigma_+)}{\sigma_+}) & \text{if } \sigma_- > 1. \end{cases}$$

Since for $\omega > 0$, $M_\omega(\sigma) \leq \sigma$, we see that under every renormalization, s_- is multiplied by a term greater than 1, and s_+ is multiplied by a term less than 1. This can be expanded upon. If $z_n = -$, then we can tell that $s_-(f_n)$ is uniformly bounded, while if $z_n = +$, then the same holds true for $s_+(f_n)$.

Thus, assume that there exists a uniform $C > 1$ such that the slopes $s_{\pm}(f_{n_k})$ are bounded between $1/C$ and C. We will now construct a sufficiently small $\delta > 0$ so that $(\underline{\sigma}, \underline{z})$ is δ-bounded.

1) Suppose that $z_n = -$. Then $s_-(f_n) < C$, i.e., $(\sigma_n - 1)\dfrac{1-\sigma_+(f_n)}{\sigma_+(f_n)} < C$. Then by Lemma 8,

$$\sigma_-(f_{n-1}) = M_{\omega_-(f_n)}(\sigma_n)\frac{1-\sigma_+(f_n)}{-\sigma_+(f_n)}.$$

It follows since $\omega_-(f_n) > 0$ that one can find $\delta_1 > 0$ small enough that if $\sigma_n < \delta_1$, then

$$\sigma_-(f_{n-1}) < \frac{\delta_1 C}{1-\delta_1} < 1 - \delta_1.$$

Using in addition that $s_-(f_n) > 1/C$ and $\omega_-(f_n) < K$, we can find $\delta_2 > 0$ sufficiently small that if $\sigma_n > 1 - \delta_2$, then

$$\sigma_-(f_{n-1}) > \frac{1-\delta_2}{C\delta_2(\delta_2 e^{K/2} + 1 - \delta_2)} > 1.$$

In this case, by making δ_2 even smaller, we obtain by Lemma 8,

$$\begin{aligned}\sigma_+(f_{n-1}) &= \sigma_+(f_n) + (1-\sigma_+)M_{\omega_-(f_n)}(\sigma_n)) \\ &> (1-\sigma_+(f_n))(-C\delta_2 + \frac{1-\delta_2}{\delta_2 e^{K/2} + 1 - \delta_2}) \\ &> \delta_2.\end{aligned}$$

For $\delta_- = \min\{\delta_1, \delta_2\}$, we obtain that δ_--boundedness cannot be broken for all n such that $z_n = -$.

2) Suppose that $z_n = +$. Then $s_+(f_n)$ is uniformly bounded, i.e.,

$$1/C < \sigma_n \frac{\sigma_-(f_n)}{\sigma_-(f_n) - 1} < C.$$

Our first goal is to find $\delta_3 > 0$ sufficiently small that $\sigma_n > 1 - \delta_3$ implies $\sigma_+(f_{n-1}) > \delta_3$. By Lemma 8,

$$\begin{aligned}\sigma_+(f_{n-1}) &= \frac{\sigma_-(f_n)M_{\omega_+(f_n)}(\sigma_n) - 1}{\sigma_-(f_n) - 1} = 1 - \frac{\sigma_-(f_n)(1 - M_{\omega_+(f_n)}(\sigma_n)}{\sigma_-(f_n) - 1} \\ &> 1 - \frac{(1 - M_{\omega_+(f_n)}(\sigma_n))C}{\sigma_n}.\end{aligned}$$

As $\omega_+(f_n) < K$, then the claim follows from the final inequality.

Next, we find $\delta_4 > 0$ sufficiently small that $\sigma_n < \delta_4$ implies $\sigma_+(f_{n-1}) < \delta_4$ or $\sigma_-(f_{n-1}) < 1 - \delta_4$. Note that since $\omega_+(f_n) > 0$,

$$\sigma_+(f_{n-1}) = 1 - \frac{(1 - M_{\omega_+(f_n)}(\sigma_n))\sigma_-(f_n))}{\sigma_-(f_n) - 1} < \sigma_n,$$

which proves our first claim. For the second claim, note that

$$\sigma_-(f_{n-1}) < C\sigma_n(\sigma_-(f_n) - 1),$$

so it suffices to check that for σ_n sufficiently small, $\sigma_-(f_n)$ is uniformly bounded. Note that $s_-(f_{n-1}) < C$, so as $\omega_+(f_n) > 0$, by the renormalization formulas for slopes,

$$C > s_-(f_{n-1}) > s_-(f_n)s_+(f_n) = \sigma_-(f_n)(1 - \sigma_n).$$

Hence, there exists δ_4 sufficiently small that $\sigma_-(f_n)$ is uniformly bounded, in which case $\sigma_-(f_{n-1})$ is bounded by a uniform multiple of δ_4, which was what we wanted. Once we set $\delta_+ = \min\{\delta_3, \delta_4\}$ and $\delta = \min\{\delta_-, \delta_+\}$, we see that the sequence $(\underline{\sigma}, \underline{z})$ is δ-bounded. By Möbius symmetry, the same conclusion holds when $K < 0$. □

Lemma 15. *Let $\mathcal{S}$ denote the right shift operator. Then $\mathcal{S}$ preserves $W_{K,\delta}$. Moreover, on W_K, $\mathcal{S}$ is the inverse of renormalization.*

Proof. Since all weights in $\mathcal{S}\underline{f}$ appear in $\underline{f}$, then it is apparent that they are uniformly bounded. It remains to check that the right shift of a δ-bounded renormalization scheme is δ-bounded. As both of the conditions for δ-boundedness depend only on the order of the renormalization scheme, and the order is preserved by $\mathcal{S}$, the claim follows.

Now suppose that $\underline{f} \in W_K$. Recalling that $\mathcal{R}f_n = f_{n-1}$, it is evident that $\mathcal{R}(\mathcal{S}\underline{f}) = \underline{f}$. In addition, if f is renormalizable, then one can construct the sequence $(\mathcal{R}\underline{f})_{\underline{z}}$ for which $\mathcal{R}f_n = f_{n-1}$ and $z_n = \text{sign}(\ln(\sigma_-(f_{n-1})))$. Note that as there are no restrictions on $\sigma_z(f)$, it is only certain that $\mathcal{R}\underline{f}_{\underline{z}}$ lies in W_K, but when it exists, it is also evident that $\mathcal{S}(\mathcal{R}\underline{f})_{\underline{z}} = \underline{f}$. This completes the proof. □

From the way we have defined W_K, it doesn't follow that every sequence in W_K is renormalizable.

5.3 Distortion Graphs and their Metric

Definition 34. *We let Γ denote the space of* ***distortion graphs*** *$\gamma : W \to \mathcal{M} \times \mathcal{M}$ such that $\pi(\gamma(\underline{f})) = (\omega_-(f), \omega_+(f))$ and $||\gamma(W_K)||$ is uniformly bounded for all $K \in \mathbb{R}$. We endow Γ with the metric*

$$||\gamma_1 - \gamma_2|| = \sup_{\underline{f} \in W} ||\gamma_1(\underline{f}) - \gamma_2(\underline{f})||.$$

Lemma 16. *Γ is a Banach space.*

Proof. Note that Γ is the space of maps into the Banach space $\mathcal{M} \times \mathcal{M}$, endowed with the uniform distance, so it is itself complete, hence a Banach space. □

5.4 Graph Transforms of Distortion Graphs

Definition 35. *Let $\gamma \in \Gamma$. We define the* ***graph transform*** *to be the operator $\mathcal{G} : \Gamma \to \Gamma$ given by*

$$\mathcal{G}\gamma(\underline{f}_{\underline{z}}) = \mathcal{R}(\underline{\sigma}, \underline{z})\gamma(\mathcal{S}\underline{f}_{\underline{z}}),$$

where $(\underline{\sigma}, \underline{z})$ denotes the renormalization scheme of $\underline{f}_{\underline{z}}$. Furthermore, we define the ***fast graph transform*** *to be the operator $\mathcal{G}_{\text{fast}} : \Gamma \to \Gamma$ given by*

$$\mathcal{G}_{\text{fast}}\gamma(\underline{f}_{\underline{z}}) = \mathcal{R}_{\text{fast}}(\underline{\sigma}, \underline{z})\gamma(\mathcal{S}^{n_1}(\underline{f}_{\underline{z}})),$$

where n_1 is the first index of the fast renormalization subsequence of $(\underline{\sigma}, \underline{z})$.

Remark.

$$\mathcal{G}^k\gamma(\underline{f}_{\underline{z}}) = \mathcal{R}^k(\underline{\sigma}, \underline{z})\gamma(\mathcal{S}^k(\underline{f}_{\underline{z}})),$$
$$\mathcal{G}^k_{\text{fast}}\gamma(\underline{f}_{\underline{z}}) = \mathcal{R}^k_{\text{fast}}(\underline{\sigma}, \underline{z})\gamma(\mathcal{S}^{n_k}(\underline{f}_{\underline{z}})).$$

5.5 Contraction of the Fast Graph Transform on $W_{K,\delta}$

Definition 36. *Let $\Gamma_{K,\delta}$, Γ_K denote the spaces of graphs in Γ restricted to $W_{K,\delta}$ and W_K, respectively.*

Remark. $\mathcal{G}, \mathcal{G}_{\text{fast}}$ *preserve* $\Gamma_{K,\delta}$.

Lemma 17. $\mathcal{G}^3_{\text{fast}}$ *is a contraction on the space of uniformly bounded graphs in* $\Gamma_{K,\delta}$.

Proof. Fix $\Gamma_{K,\delta}$. Let $\gamma_1, \gamma_2 \in \Gamma_{K,\delta}$ be uniformly bounded by $L > 0$. Then, by the Contraction Theorem, there exists $\kappa < 1$ depending only on L and δ such that for any $\underline{f} \in W_{K,\delta}$ with renormalization scheme $(\underline{\sigma}, \underline{z})$,

$$\begin{aligned}||\mathcal{G}^3_{\text{fast}}\gamma_1(\underline{f}) - \mathcal{G}^3_{\text{fast}}\gamma_2(\underline{f})|| &= ||\mathcal{R}^3_{\text{fast}}(\underline{\sigma}, \underline{z})\gamma_1(\mathcal{S}^{n_3}\underline{f}) - \mathcal{R}^3_{\text{fast}}(\underline{\sigma}, \underline{z})\gamma_2(\mathcal{S}^{n_3}\underline{f})|| \\ &< \kappa||\gamma_1(\mathcal{S}^{n_3}\underline{f}) - \gamma_2(\mathcal{S}^{n_3}\underline{f})|| < \kappa||\gamma_2 - \gamma_1||,\end{aligned}$$

since $\gamma_1(\mathcal{S}^{n_3}\underline{f}), \gamma_2(\mathcal{S}^{n_3}\underline{f})$ are in the same slice of $\mathcal{M} \times \mathcal{M}$. Thus, $||\mathcal{G}^3_{\text{fast}}\gamma_1 - \mathcal{G}^3_{\text{fast}}\gamma_2|| < \kappa||\gamma_1 - \gamma_2||$, which was what we wanted. □

5.6 Convergence Theorem

Theorem 2 (Convergence Theorem)**.** *There exists a $\mathcal{G}$-invariant graph in Γ, i.e., over every $\underline{f}$ in the unstable manifolds of renormalization lies an $\mathcal{R}$-invariant distortion measure.*

Proof. Note that for any choice of K, δ, $\Gamma_{K,\delta}$ is a complete space, upon which by Lemma 17 the fast graph transform acts as a contraction in three steps. Moreover, by Lemma 14, the distortion measure pairs $\gamma(\underline{f})$ remain uniformly bounded after application of the fast graph transform, so there is a uniform rate κ of contraction for uniformly bounded γ. Hence, being a contraction on a complete space, there exists a unique fixed point of $\mathcal{G}_{\text{fast}}$ in $\Gamma_{K,\delta}$. Letting δ range over $(0, 1/2)$, we obtain a fixed point on Γ_K. And finally, since Γ is the disjoint union of the spaces Γ_K, we obtain a unique fixed point for Γ. Finally, we can use that $\mathcal{G}$ doesn't expand distances to prove that the fixed point for $\mathcal{G}_{\text{fast}}$ is also a fixed point for $\mathcal{G}$. As the unstable manifolds of renormalization for Ω can be embedded into W, it follows that they have an associated $\mathcal{R}$-invariant distortion measure pair. □

6 Properties of Limiting Distortion Measures

6.1 Monotonicity Properties

Lemma 18. *Let (μ_-, μ_+) be the $\mathcal{R}$-invariant measure associated to $\underline{f} \in W_{K,\delta}$. Then:*
1) *If $K = 0$, then (μ_-, μ_+) is the zero measure.*

2) *If $K \neq 0$, then (μ_-, μ_+) is either strictly negative, or strictly positive, with the same sign as K.*

Proof. 1) Note that all renormalization operators preserve the zero measure pair $(0,0)$, so by uniqueness of he $\mathcal{G}$ fixed point of Γ_0, $(\mu_-, \mu_+) = (0,0)$.
2) Suppose that $K \neq 0$. Observe that if (ν_-, ν_+) are either both strictly positive or both strictly negative, then that property is preserved by any renormalization operator $\mathcal{R}_z^\sigma$. Hence, this property is also preserved by $\mathcal{G}$, so the $\mathcal{G}$ fixed point of Γ_K, which can be attained as the limit of $\mathcal{G}$ on such strictly monotone distortion measure pairs, must share that monotone property. It just remains to remark that if (μ_-, μ_+) are either both strictly positive, or both strictly negative, and lie in $(\mathcal{M} \times \mathcal{M})_K$, then μ_-, μ_+ share the same sign as K. $\square$

6.2 Results for Bounded Combinatorics

6.2.1 Definition of (N, δ)-bounded Combinatorics

Definition 37. *For $N \in \mathbb{N}$ and $\delta \in (0, 1/2)$, a renormalization scheme $(\underline{\sigma}, \underline{z})$ is said to be (N, δ)-bounded if the sign of z_n changes at most in every $N+1$ steps and $\sigma_n \in (\delta, 1-\delta)$ for all $n \in \mathbb{N}$.*

Remark. *(N, δ)-bounded renormalization schemes correspond to bounded combinatorics of Möbius circle maps.*

6.2.2 Dyadic Doubling of (N, δ)-bounded Distortion Measures

Definition 38. *A measure μ defined on $\mathcal{A}$ is said to be **dyadic doubling** if the measure of any two standard dyadic intervals of scale n that share a common dyadic ancestor at scale $n-1$ are proportional by a constant independent of n.*

Remark. *If μ is dyadic doubling, then μ is also bi-Hölder.*

Proposition 8. *Suppose that the renormalization scheme $(\underline{\sigma}, \underline{z})$ is (N, δ)-bounded. Then for any $\underline{f} \in W_K$ with the same renormalization scheme, the corresponding $\mathcal{R}$-invariant distortion measure pair of $\underline{f}$ is composed of dyadic doubling measures.*

Proof. Let $(\underline{\sigma}, \underline{z})$ be an (N, δ)-bounded renormalization scheme associated to $\underline{f} \in W_{K,\delta}$. Since $\sigma_n \in (\delta, 1-\delta)$ for all $n \in \mathbb{N}$, it is trivial to check that an (N, δ)-bounded renormalization scheme is δ-bounded. Let (μ_-, μ_+) be the invariant distortion measure pair associated to $\underline{f}$. Note that $\mathcal{S}^n \underline{f}$ also has an associated invariant distortion measure pair with an (N, δ)-bounded renormalization scheme for all $n \in \mathbb{N}$. We will first find a lower bound for $\mu_\pm(0,1]$, which will provide us with the first proportional estimates. As an upper bound, by Lemma 18, $|\mu_-(0,1]|, |\mu_+(0,1]| \leq |K|$. Without loss of generality, suppose that $z_1 = -$. By the Zoom Contraction Law, there exists $\kappa < 1$ depending only on δ and $|K|$ such that $|\mu_-(0,1]| < \kappa |K|$, so $|\mu_+(0,1]| > (1-\kappa)|K|$. Now suppose that $n_1 \leq N$ (because the renormalization scheme is (N, δ)-bounded) is the first index of the fast renormalization subsequence of $(\underline{\sigma}, \underline{z})$. Assume that $(\mu_{1,-}, \mu_{1,+})$ are the associated invariant measures of $\mathcal{S}^{n_1} \underline{f}$. By symmetry, as $z_{n_1+1} = +$, we obtain that $|\mu_{1,-}(0,1]| > (1-\kappa)|K|$. To get back to (μ_-, μ_+), we apply the operator sequence $\mathcal{R}_-^{\sigma_1} \circ ... \circ \mathcal{R}_-^{\sigma_{n_1}}$, which acts on $\mu_{1,-}$ as a zoom to an interval of length

$\sigma_1\sigma_2...\sigma_{n_1}$, which has size bounded above by δ^N. By the Zoom Contraction Law applied to the complementary interval, there exists $\tilde{\kappa} < 1$ depending only on $|K|, 1-\delta^N$ such that $\mu_-(0,1] > (1-\tilde{\kappa})(1-\kappa)|K|$, which was what we wanted. By symmetry, if $z_1 = +$, then $|\mu_-(0,1]| > (1-\kappa)|K|, |\mu_+(0,1]| > (1-\tilde{\kappa})(1-\kappa)|K|$. This proves the desired lower bounds.

The next step is a technical result. It claims that if $\omega_2 = C\omega_1$ for $C > 1, |\omega_2| < 4$ and if $I \subseteq [0,1]$ is a subinterval, then

$$\frac{\int_I \eta_{\omega_2}}{\int_I \eta_{\omega_1}} < \frac{C}{1-|\omega_2|/4}.$$

To prove this, note that the ratio $\dfrac{\eta_{\omega_2}(x)}{\eta_{\omega_1}(x)}$ is bounded above by $\dfrac{\eta_{|\omega_2|}(1)}{\eta_{|\omega_1|}(1)}$, since the derivative $D\eta_\omega(x)$ equals $[\eta_\omega(x)]^2/2$, which is strictly monotone and increases with $|\eta_\omega(x)|$. Hence, by using power series expansions,

$$\begin{aligned}\frac{\int_I \eta_{\omega_2}}{\int_I \eta_{\omega_1}} &< \frac{e^{|\omega_2|/2}-1}{e^{|\omega_1|/2}-1} < \frac{C|\omega_1|/2 + (C|\omega_1|/2)^2/2! + ...}{|\omega_1|/2} \\ &< \sum_{i=0}^{\infty} C(|\omega_2|/4)^i = \frac{C}{1-|\omega_2|/4},\end{aligned}$$

where we utilized that $|\omega_2|/4 < 1$ to write the geometric sequence in closed form. This completes the proof of the claim.

By the Zoom Contraction Law, any time a distortion measure is cut by a zoom operator, its size shrinks at least by a factor of $\kappa < 1$, where κ depends on $|K|$ and δ. Thus, for some finite integer k depending on $|K|$ and κ, if the mass $\mu_\pm(\tau_1,\tau_2]$ was obtained by applying k zoom operators to one of the invariant measures associated to $\mathcal{S}^m\underline{f}$, then $\mu_\pm(\tau_1,\tau_2] < 1$. Keep in mind that for any pair $(z,(\tau_1,\tau_2])$ of a sign with a standard dyadic interval of scale $n \geq 1$ (corresponding to $\mu_z(\tau_1,\tau_2]$), there exists a unique pair $(\tilde{z},(\tilde{\tau_1},\tilde{\tau_2}])$, where $(\tilde{\tau_1},\tilde{\tau_2}]$ is a standard dyadic interval of scale either n or $n-1$, for which $\mu_z(\tau_1,\tau_2]$ either equals, or was obtained as a zoom from $\mu_{1,\tilde{z}}(\tilde{\tau_1},\tilde{\tau_2}]$, where $\mu_{1,\pm}$ are the invariant measures associated to $\mathcal{S}\underline{f}$. This process of selecting ancestors can be iterated backwards in time, until an interval of unit size is obtained. As the sign of the renormalization operator changes at most every $N+1$ steps, it follows that an interval must have been zoomed at least once every $N+1$ steps during this sequence. Consequently, any sufficiently small standard dyadic interval, say of size 2^{-m}, has measure bounded by 1, and this holds true for any of the invariant distortion measures $\mu_{k,\pm}$ associated to $\mathcal{S}^k\underline{f}$. Note that the selection of ancestors sends pairs of standard dyadic intervals of scale $n > 1$ with common ancestors to pairs that satisfy the same property. As a result, the proportions between measures of standard dyadic intervals sharing the same ancestor can only change by means of a zoom operation. We observe that if T_{n-1} is the ancestor of the standard dyadic interval T_n, and $\mu_\pm(T_{n-1}) = C\mu_\pm(T_n)$ for $C > 1$, then if $1 > |I| > \delta > 0$, it follows that

$$\frac{Z_I\mu_\pm(T_{n-1})}{Z_I\mu_\pm(T_n)} < \frac{C}{1-\kappa},$$

because Z_I must preserve at least $1-\kappa$ of the original mass of T_n and $|Z_I\mu_\pm(T_{n-1})| < \mu_\pm(T_{n-1})$. Applying this claim for the first k zooms backwards in time, and the previous claim for all zooms afterwards, it would suffice to show that if $\tilde{T}$ is the dyadic ancestor of a standard dyadic interval T, then $\mu_{\pm}(T)$ is uniformly proportional to $\mu_{\pm}(\tilde{T})$. Since the measure of intervals decays exponentially to 0, and it takes only k zooms to get to intervals of scale m, for which the proportion decays at most by $(1-\kappa)$, it follows that the claim holds true whenever the sequence of ancestors share left endpoints, since then the zoom operators are always the same, and the proportion of weights can only shrink from that point by the exponentially decaying factors $1-\mu_{\pm}(\tilde{T})/4 > 3/4$. If $\tilde{T}$ and T share a right endpoint, then the zoom interval is transformed by a Möbius transformation of weight $\mu_{\pm}(T_1)$, where T_1 is the standard dyadic interval complementary to T whose ancestor is $\tilde{T}$. When T_1 is of of sufficiently bounded scale, the interval shrinks by at most a factor of $1-|\mu_{\pm}(T_1)|$ which exponentially decays to 1. Thus, the claim holds for all dyadic successors T of $\tilde{T}$, which was what we wanted. □