MAT 312 Applied Algebra
Summer II 2009
Practice midterm

(1) Find the gcd and lcm of 12, 42, 81.

Solution: *$12 = 2^2 \times 3$, $42 = 2 \times 3 \times 7$, $81 = 3^4$, hence $(12, 42, 81) = 2 \times 3 = 6$, and $lcm(12, 42, 81) = 324$.*

(2) Compute $[17]_{20}^5$ and $[13]_9^{-2}$

Solution: $[17]_{20}^5 = [-3]_{20}^5 = [81]_{20}[-3]_{20} = [-3]_{20}$

(3) Solve $9x \equiv 12 \mod 15$.

Solution: *Since $(9, 15) = 3$, we can divide both sides by 3 to get*

$$3x \equiv 4 \mod 5.$$

Since $(3, 5) = 1$, there is a unique solution, say x_0, then $x_0 = [3]_5$, hence all three solutions to $9x \equiv 12 \mod 15$ are

$$[3]_{15}, [8]_{15}, [13]_{15}$$

(4) Solve

$$\begin{aligned} 6x &\equiv 4 \mod 10 \\ x &\equiv 12 \mod 8 \end{aligned}$$

Solution: *Simplifying the first equation, we get*

$$\begin{aligned} 3x &\equiv 2 \mod 5 \\ x &\equiv 12 \mod 8 \end{aligned}$$

since $(5, 8) = 1$, by Chinese Remainder theorem, there is a unique solution. Consider

$$5 \times 5 - 8 \times 3 = 1,$$

define

$$x := 2 \times 8 \times (-3) + 12 \times 5 \times 5 = 252$$

then $[252]_{40} = [12]_{40}$ *is the unique solution.*

(5) What is the last digit of 3^{1023}?

Solution: *Since* $3^4 \equiv 1 \mod 10$, *and* $1023 \equiv 3 \mod 4$, *we have*

$$3^{1023} \equiv 3^3 \mod 10 \equiv 7 \mod 10.$$

Hence the last digit of 3^{1023} *is 7.*

(6) Let p be a prime number, show that $x^2 \equiv 4 \mod p$ has just two solutions in $\mathbb{Z}_p$

Proof: *Since* $x^2 - 4 = (x+2)(x-2)$ *and* p *is prime, if* $x^2 \equiv 4 \mod p$, *then* $p|(x+2)$ *or* $p|(x-2)$, *hence* $x = [2]_p$ *or* $[-2]_p$.

(7) Find the inverse of $f(x) = (x^3 - 3x^2 + 3x - 1)/8$.

Solution: $f(x) = (x-1)^3/8 = (\frac{x-1}{2})^3 = y$, *then*

$$x = 1 + 2y^{1/3},$$

hence the inverse of $f(x)$ *is* $f^{-1}(x) = 1 + 2x^{1/3}$