# Research Statement
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My research interests lie within the intersection of Algebraic Topology and Algebraic Geometry. In my thesis I construct algebraic cocycles which generalize the Gauss map for projective hypersurfaces. I study their geometric properties and I relate them to the cocycle determined by the classical Gauss map in the case of smooth hypersurfaces. Another set of results in my thesis concerns the construction of pairings in the space of algebraic cycles in $\mathbb{P}^n$. These pairings are related to the map which classifies the tensor product of vector bundles. I construct such a pairing for codimension 1 cycles and I find topological obstructions for the existence of a general pairing. All these results are explained in section 2 below. In section 1 I give some general background.

## 1 General Framework

In the 1950's algebraic topologists realized that the homology and cohomology groups can be represented as homotopy classes of maps from one space into another. Dold and Thom constructed in [DT58] particularly beautiful models for the Eilenberg-MacLane spectrum. Their representations of homology and cohomology were

$$H_i(X, \mathbb{Z}) = [\mathbb{S}^i, \mathbb{Z} \cdot X] = \pi_i(\mathbb{Z} \cdot X) \qquad H^i(X, \mathbb{Z}) = [X, \mathbb{Z} \cdot \mathbb{S}^i]$$

where $X$ is any compact, connected, CW-complex and $\mathbb{Z} \cdot X$ is the free abelian group generated by the points of $X$.

From the point of view of algebraic geometry, the free abelian group can be interpreted as the group of 0-dimensional cycles, thus it is natural to consider the group of $p$-dimensional algebraic cycles $\mathcal{Z}_p(X)$. This is something which Blaine Lawson did in his foundational paper [LJ89]. The consequences of these investigations have been far reaching for both the study of invariants of algebraic varieties and the study of the geometry of spectra. The following is a list of some of the results derived from these considerations:

1. There are covariant functors $L_pH_k$ which associate a group to any projective algebraic variety
$$L_pH_k(X) = \pi_{k-2p}\mathcal{Z}_p(X)$$
These functors agree with singular homology $H_k(X, \mathbb{Z})$ when $p = 0$ and with the group of cycles modulo algebraic equivalence $\mathcal{A}(X)$ when $2p = k$. Hence, we may think of these functors as a family which captures topological information when $p = 0$ and becomes more algebraic as $p$ increases. What happens between the two extreme cases is intriguing. For example, Griffiths proved in [Gri69] that there exist threefolds for which $\mathcal{A}(X)$ is infinitely generated, whereas singular homology is always finitely generated for a compact CW-complex. Eric Friedlander coined the term *Lawson Homology* for the groups $L_pH_k$. These functors come equipped with natural transformations into singular homology. There are relative versions of these functors and generalizations to quasi-projective varieties and varieties with symmetries [LJ95].

2. There are contravariant functors $L^pH^k$ (called morphic cohomology) which analogously generalize singular cohomology. These functors arise from considering the space of algebraic morphisms from $X$ into the space of cycles in $\mathbb{P}^n$. Namely
$$L^pH^k(X) = \pi_{2p-k}(\mathcal{Z}^p(X; \mathbb{C}^n)) = \pi_{2p-k}\mathcal{M}or(X, \mathcal{C}^p(\mathbb{P}^\infty))^+/\mathcal{M}or(X, \mathcal{C}^p(\mathbb{P}^\infty)^+$$
More generally, there is a bivariant group valued functor $L^pH^k(X, Y)$ of *algebraic cocycles in $X$ with values in $Y$* [FLJ92].

   There is a generalized Poincaré duality statement: If $X$ is a smooth $n$-dimensional projective variety, then $L^pH^k(X) \cong L_{n-p}H_{n-k}(X)$ [FLJ97].

3. The inclusion of the cycles of degree 1 (i.e. linear subspaces) in $\mathbb{P}^n$ into the space of all cycles stabilizes to give a map $c : BU \to \mathcal{Z}(\mathbb{P}^\infty)$ which represents the total chern class map from topological $K$-theory into the cohomology ring $1 \times \prod_{k\geq 0} H^{2k}$ thought of as a group with respect to the cup product pairing. The existence of such a map was first observed by Grothendieck, and it was conjectured by Segal in [Seg75] that this map extends to a map of cohomology theories. In [BLLMM93] this conjecture was settled by showing that the map $c$ is actually a map of $E_\infty$-spectra.

A beautiful recount of the results up to 1995 is given in [LJ95].

This study of the homotopy type of the spaces of algebraic cycles has continued to expand recently in different directions. Chris Peters calculated the Lawson Homology of varieties with small Chow groups in [Pet00]. Last year, Wenchuan Hu proved that for a projective smooth variety of dimension $n$, $L_1H_k$ and $L_{n-2}H_k$ provide new birational invariants.

The relation of morphic cohomology with motivic cohomology and $K$-theory has been extensively studied by Eric Friedlander, Mark Walker, et al. They defined a theory called *semi-topological $K$-theory* which lies between algebraic $K$-theory and topological $K$-theory. This theory has a chern class map and a natural map into topological $K$-theory which is compatible with the cycle class map (see [FW02, FHW04, CLF01]).

The results related to the Segal conjecture have been generalized by Lawson, Lima-Filho and Michelson in [LJLFM96]. Paulo Lima-Filho has constructed families of spectra $K_X$ and $\mathcal{M}_X$ parametrized by algebraic varieties $X$ which have generalized total chern class maps which are maps of $E_\infty$-spectra (see [LF99]).

# 2 Results obtained

My dissertation research is divided into two parts

- Finding algebraic cocycles which are geometrically meaningful.
- Studying pairings of spectra related to the tensor product on $K$-theory.

## 2.1 Geometry of algebraic cocycles

An effective algebraic $p$-cocycle on a projective variety $X$ with values in a quasi projective variety $Y$ is an algebraic mapping from $X$ into the space of algebraic cycles $\mathcal{C}^p(Y)$ of codimension $p$ in $Y$. Algebraic cocycles with values in $\mathbb{C}^n$ define elements in morphic cohomology and therefore in singular cohomology via natural transformations. For example, given a flat map of projective varieties $f : X \to Y$, we obtain an algebraic cocycle on $X$ with values in $Y$, namely the cocycle $x \mapsto f^{-1}(x)$.

Another example is the classifying map of an algebraic vector bundle generated by its global sections

$$X \to \mathcal{G}^p(\mathbb{P}^n)$$

where $\mathcal{G}^p(\mathbb{P}^n)$ is the grassmannian of codimension $p$ linear spaces in $\mathbb{P}^n$ (i.e. cycles of degree 1). This map defines an effective algebraic cocycle with values in $\mathbb{P}^n$ which represents the total chern class of the bundle (in both morphic cohomology and singular cohomology). My results deal with the construction of similar maps with values in projective cycles of higher degree.

### 2.1.1 Higher Degree Gauss Maps

For any smooth variety $X \subset \mathbb{P}^n$ of codimension $q$ the classical gauss map is the map

$$g^1 : X \to \mathcal{G}^q(\mathbb{P}^n)$$

which associates to each point $\xi$ the projective linear subspace of codimension $q$ tangent to $X$ at $\xi$ in $\mathbb{P}^n$:

$$\xi \mapsto \overline{T_\xi X}$$

If $X$ is a hypersurface defined by the set of zeros of a homogeneous polynomial $F$, $X = V(F) := \{x \in \mathbb{P}^n \mid F(x) = 0\} \subset \mathbb{P}^n$, then the gauss map has the following coordinate expression

$$\xi \mapsto V\left(\sum \frac{\partial F}{\partial x_i}(\xi) x_i\right)$$

If $X$ has singularities we no longer have a map which is regular, but only a rational map.

In my thesis I construct higher degree Gauss maps

$$g^k : X \to \mathcal{C}^1_k(\mathbb{P}^n)$$

which associate to each point $\xi$ the effective algebraic cycle of degree $k$ and codimension 1 which best approximates $X$ at $\xi$. In analogy with the above, the degree $k$ Gauss map associates to the point $\xi$ a projective hypersurface defined by the $k$-th partial derivatives of $F$ at $\xi$.

As in the case of the first Gauss map, the higher degree Gauss maps are only rational in general. Interestingly however, they can still be regular in the presence of certain singularities. More precisely, the following is true:

**Theorem 2.1.** *If a hypersurface of degree $d$ has a regular Gauss map of degree $p$, it also has regular Gauss maps of degree $q$ for $p \leq q \leq d$*

**Remark 2.2.** These higher degree Gauss maps are **not** the higher order Gauss maps which define the Higher Fundamental Forms studied by Griffiths and Harris, Landsberg et. al.

**Example 2.3.** Let $V \subset \mathbb{P}^2$ be the nodal plane cubic defined by $F(x_0, x_1, x_2) = x_2x_1^2 - x_0^3 - x_0^2x_2$, then $V$ does not have a regular Gauss map of degree 1, but it has a well defined Gauss map of degree 2 (see figure 1):

$$\overline{\xi} \mapsto V_{\overline{\xi}} = V(-(3\xi_0 + \xi_2)x_0^2 - 2\xi_0x_0x_2 + \xi_2x_1^2 + 2\xi_1x_1x_2)$$

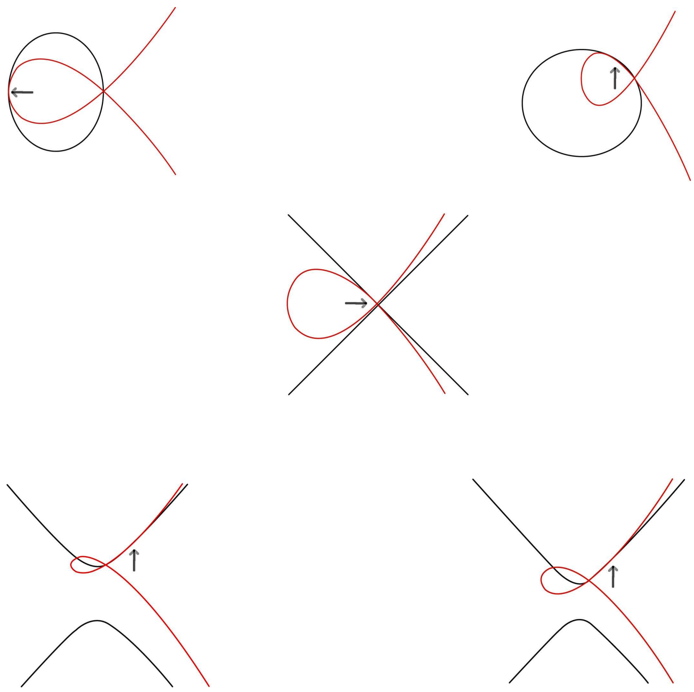

Figure 1: Second degree approximations to the nodal cubic: The red curve is the nodal cubic, the black curves are the conics approximating the curve at the point signaled by the arrow.

That is, to every point $v$ we associate a quadric which approximates the curve at $v$. Notice that at the node $[0 : 0 : 1]$ we do have a well defined second order approximation: the union of the two possible tangents.

The closure of the image of the first Gauss map defines the dual variety of the hypersurface $X$. We could ask what are the general properties of the images of these higher degree Gauss maps. The following results extend some classic results of projective geometry.

**Theorem 2.4.** *If $X$ is not a cone, then the $(d-1)$ Gauss map $g^{(d-1)}$ is an isomorphism from $X$ into its image. (This extends the well known result for the duals of smooth quadrics. Note that any singular quadric is a cone.)*

**Theorem 2.5.** *Generically, the degree of the $p$-th Gauss image variety is $d(d-p)^{n-1}$. (This extends the classical formulas for the degree of the dual of a smooth hypersurface).*

As I mentioned in the introduction, the higher degree Gauss maps define classes in cohomology. Carefully defining the structure of $H$-space in $\mathbb{P}^\infty$ it is possible to obtain the following result:

**Theorem 2.6.** *If $X$ is a smooth hypersurface, the cohomology class defined by the $p$-th gauss map $[g^p] \in H^2(X)$ satisfies*

$$[g^p] = \frac{d-p}{d-1}[g^1] = \frac{d-p}{d-1}c(N_X(-1))$$

This last theorem is proved by constructing explicit homotopies between the right side and the left side of the first equality (the second equality is a classical result). The homotopy describes a relation between the tensor product operation in $H^2$ and the procedure of taking partial derivatives of the homogeneous coordinates of the classifying map.

## 2.2 Pairings of spectra extending the tensor product of vector bundles

Boyer, Lawson, Lima-Filho, Mann and Michelson settled the Segal conjecture in [BLLMM93]. One of the fundamental results which motivated the proof is that there is a geometric construction which extends the map classifying the direct sum of vector bundles in BU to the space $\mathcal{Z}$ of all algebraic cycles, namely, the linear join # of cycles, i.e. the following diagram commutes:

$$\begin{array}{ccc} \mathrm{BU} \times \mathrm{BU} & \xrightarrow{\oplus} & \mathrm{BU} \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} \\ \mathcal{Z} \times \mathcal{Z} & \xrightarrow{\#} & \mathcal{Z} \end{array}$$

In this paper the authors mention that one would like to have a geometric construction on the space of algebraic cycles which extends the tensor product in the level of BU (i.e. degree one cycles). Segal proved in [Seg74] that BU has an infinite loop space structure where the $H$-space structure is induced by the map classifying the tensor product of vector bundles. Therefore the construction requested by the authors of [BLLMM93] would give yet another infinite loop space structure on $\mathcal{Z}$. My results provide an idea of the extent to which such construction is possible:

**Theorem 2.7.** *There is an algebraic biadditive pairing $\hat{\otimes}$ which extends the tensor product to all effective divisors:*

$$\mathcal{C}^1_d(\mathbb{P}^n) \times \mathcal{C}^1_e(\mathbb{P}^m) \xrightarrow{\hat{\otimes}} \mathcal{C}^1_{de}(\mathbb{P}^{mn+m+n})$$

This product is constructed via an algebraic pairing in the corresponding rings of polynomials which may be of interest in its own right. The formula obtained from stabilizing and group-completing the pairing to the stabilized space $\mathcal{Z}^1_0(\mathbb{P}^\infty)$ of algebraic cycles of codimension 1 and degree 0 yields a commutative diagram

$$\begin{array}{ccc} \mathcal{G}^1(\mathbb{P}^\infty) \times \mathcal{G}^1(\mathbb{P}^\infty) & \longrightarrow & \mathcal{G}^1(\mathbb{P}^\infty) \\ \downarrow c_1 & & \downarrow c_1 \\ \mathcal{Z}^1_0(\mathbb{P}^\infty) \times \mathcal{Z}^1_0(\mathbb{P}^\infty) & \xrightarrow{\hat{\otimes}} & \mathcal{Z}^1_0(\mathbb{P}^\infty) \end{array}$$

which recovers the group structure in the second cohomology group given by the tensor product of line bundles

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) \in H^2(BU_1)$$

The Hurewicz map is the main tool in proving that a general pairing does not exist. The following theorem calculates the classes pulled back by the Hurewicz map.

**Theorem 2.8.** *The inclusion of the grassmannian $\mathcal{G}^1(\mathbb{P}^n)$ of hyperplanes in $\mathbb{P}^n$ into the space $\mathcal{Z}^1(\mathbb{P}^n)$ of all cycles in $\mathbb{P}^n$ factors through the free group $\mathbb{Z}\mathcal{G}^1(\mathbb{P}^n)$:*

$$\begin{array}{ccccc} \mathcal{G}^1(\mathbb{P}^n) & \xhookrightarrow{i} & \mathbb{Z}\mathcal{G}^1(\mathbb{P}^n) & \xhookrightarrow{j} & \mathcal{Z}^1(\mathbb{P}^n) \\ & & \downarrow = & & \downarrow = \\ & & \prod_{j=0}^n K(\mathbb{Z}, 2j) & & K(\mathbb{Z}, 0) \times K(\mathbb{Z}, 2) \end{array} \tag{2.1}$$

*With respect to the canonical product decomposition given in (2.1), the map $i$ classifies the cohomology class $1 \times \omega \times \cdots \times \omega^n$ where $\omega$ is the multiplicative generator of $H^2(\mathcal{G}^1(\mathbb{P}^n))$ and the map $j$ is homotopic to the projection $\pi_0 \times \pi_1$*

The pairing constructed for divisors in (2.7) cannot be extended to a continuous biadditive pairing on the space of cycles of higher codimension, but it does admit an extension if we restrict the second factor of the pairing to the subgroup $\mathbb{Z}\mathcal{G}^1(\mathbb{P}^m)$ of cycles which are unions of hyperplanes (possibly with multiplicities).

**Theorem 2.9.** *There is a continuous biadditive pairing $\tilde{\otimes}$ which makes the following diagram commute*

$$\begin{array}{ccc} \mathcal{G}^p(\mathbb{P}^n) \times \mathcal{G}^1(\mathbb{P}^m) & \xrightarrow{\otimes} & \mathcal{G}^p(\mathbb{P}^{nm+n+m}) \\ \downarrow{\scriptstyle c\times h} & & \downarrow{\scriptstyle c} \\ \mathcal{Z}_0^p(\mathbb{P}^n) \times \mathbb{Z}_0\mathcal{G}^1(\mathbb{P}^m) & \xrightarrow{\tilde{\otimes}} & \mathcal{Z}_0^p(\mathbb{P}^{nm+n+m}) \end{array}$$

The relevance of this diagram is twofold, on the one hand it provides a new way of calculating the formula for the total chern class of the tensor product of a vector bundle and a line bundle, namely

$$c_i(E \otimes L) = \sum_{j=0}^{i} \binom{rk(E)-j}{i-j} c_j(E) c_1(L)^{i-j} \tag{2.2}$$

and on the other hand it suggests a path for generalizing the Bott periodicity map which is related to the top arrow of this diagram (the problem for generalizing the Bott map is that there is no "orthogonal complement" in the space of cycles.)
The formula 2.2 does not describe completely the map induced in cohomology by the pairing $\tilde{\otimes}$, since $h^*(i_{2k}^m) = h^*(i_{2t}^s) = \omega^{km}$ if $km = ts$. This calculation can actually be obtained rationally:

**Theorem 2.10.** *The pairing*

$$\mathcal{Z}_0^p(\mathbb{P}^n) \times \mathbb{Z}_0\mathcal{G}^1(\mathbb{P}^m) \to \mathcal{Z}^p(\mathbb{P}^{nm+n+m})$$

*induces the following map in rational cohomology*

$$\tilde{\otimes}^*(i_{2k}) = \sum_{j=0}^{k} \binom{p-j}{k-j} i_{2j} \otimes \tilde{i}_{2(i-j)}$$

*where $i_{2k}$ is the fundamental class of the $k$-th factor of $\mathcal{Z}_0^p(\mathbb{P}^n) \simeq \prod_{j=1}^p K(\mathbb{Z}, 2j)$ and $\tilde{i}_{2l}$ is the fundamental class in the $l$-th factor of $\mathbb{Z}_0\mathcal{G}^1(\mathbb{P}^m) \simeq \prod_{j=1}^m K(\mathbb{Z}, 2j)$*

Using theorems 2.8 and 2.10 the following theorem can be proved

**Theorem 2.11.** *There is no continuous biaddditive pairing in the stabilized space of cycles which makes the following diagram commute*

$$\begin{array}{ccc} \mathrm{BU} \times \mathrm{BU} & \xrightarrow{\otimes} & \mathrm{BU} \\ {\scriptstyle c\times c}\downarrow & & \downarrow{\scriptstyle c} \\ \mathcal{Z} \times \mathcal{Z} & \xrightarrow{\tilde{\otimes}} & \mathcal{Z} \end{array}$$

# 3 Research Plans

The interplay between geometric constructions in classifying spaces and invariants of algebraic varieties opens the possibilities for various lines of research: Extending geometric constructions on vector bundles to the space of cycles; Providing geometric constructions in the space of cycles which realize known relations in (co)homology; Finding explicit representations of interesting (co)homology classes.

This interplay is illustrated by the results in section 2: The pairing 2.7 is a geometric construction which recovers the group structure in $H^2$; The non-existence theorem 2.11 states the impossibility of a geometric construction and it is proved using topological information of the classifying spaces; The calculation of the chern class 2.6 which provides information about the dimension and the degree of a variety is obtained by constructing an explicit homotopy of the higher degree gauss maps.
The following is a list of projects on which I am currently working:

- The results obtained concerning the non-existence of a tensor product operation in the cycle space, lead naturally to consider the factorization of the total chern class map through the Hurewicz map. This factorization was also used by Totaro in [Tot95] to calculate the homology operations on the spectrum defined in [BLLMM93]. The target of this Hurewicz map is the free abelian group generated by the points of the grassmannian. This space admits the action of an operad and becomes an $E_\infty$-ring space. It would be interesting to calculate the pairings induced by the ring structure corresponding to the direct sum and the tensor product. The following result gives an example of such calculations:

  **Theorem 3.1.** *The space* $\mathbb{Z}\mathrm{BU}_1 \simeq \prod_{i=0}^{\infty} K(\mathbb{Z}, 2i)$ *is an* $E_\infty$*-ring space with the structure* $\tilde{\otimes}$ *induced by the tensor product of line bundles. In rational cohomology, the* $\tilde{\otimes}$ *pairing is determined by the formula*

  $$\tilde{\otimes}^*(i_{2k}) = \sum_{j=0}^{k} \binom{k}{j} i_{2j} \otimes i_{2(k-j)}$$

- Another project I have is to extend the higher degree Gauss maps to codimension greater than 1. The first problem is that there is no obvious definition in general. There are various constructions using Gröbner bases which might work at least in the case of a complete intersection. Also, complications arise from the fact that for cycles of codimension greater than 1 we no longer have the tensor product structure which is the basic tool for calculating the cohomology classes defined by the higher degree Gauss maps. In any event, for any polynomial in the ideal defining a variety we obtain higher degree hypersurfaces. I would like to investigate in which cases we have well defined higher degree Gauss maps of higher codimension.

- Recently I have been studying a natural family of algebraic cocycles on $\mathcal{C}^p(\mathbb{P}^n)$ with values in $\mathcal{C}^{p-k}(G_k(\mathbb{P}^n))$ induced by a certain incidence correspondence on the grassmannian. Namely, there is a family of maps $\psi_k$ for $k \leq p$ defined as follows:
$$\psi_k : \mathcal{C}^p(\mathbb{P}^n) \to \mathcal{C}^{p-k}\left(\mathcal{G}_k(\mathbb{P}^n)\right)$$
is given by
$$\eta \mapsto \{\Lambda \mid \Lambda \cap \eta \neq \varnothing\}$$
The case $k = p - 1$ is the map associating to each cycle its Chow form. This family can be thought of as a family of natural transformations between cohomology theories or as a set of cohomological invariants of the stabilized Chow variety $\mathcal{C}^p(\mathbb{P}^n)$. The maps of the family just defined take cycles of degree 1 (i.e. linear spaces) into the schubert cycles in the grassmannian. These maps are interesting for their geometric meaning and also for their behavior as mappings of spectra.

# References

[BLLMM93] Charles P. Boyer, H. Blaine Lawson, Paulo Lima-Filho, Benjamin M. Mann, and Marie-Louise Michelsohn. Algebraic cycles and infinite loop spaces. *Invent. Math.*, 113:373–388, 1993.

[CLF01] R.L. Cohen and P. Lima-Filho. Holomorphic $k$-theory, algebraic cocycles, and loop groups. *K-theory*, 23:345–376, 2001.

[DT58] Albrecht Dold and René Thom. Quasifaserungen und unendliche symmetrische produkte. *Ann. of Math.*, 2(67):230–281, 1958.

[FHW04] Eric Friedlander, Christian Haesemeyer, and Mark Walker. Techniques, computations and conjectures for semi-topological $k$-theory. *Math. Ann.*, 330, 2004.

[FLJ92] Eric Friedlander and H. Blaine Lawson Jr. A theory of algebraic cocycles. *Annals of Mathematics*, 136:361–428, 1992.

[FLJ97] Eric Friedlander and H. Blaine Lawson Jr. Duality relating spaces of algebraic cocycles and cycles. *Topology*, 36(2):533–565, 1997.

[FW02] Eric Friedlander and Mark Walker. Semi-topological $k$-theory using function complexes. *Topology*, 41(3):591–644, 2002.

[Gri69] Philip A. Griffiths. On the periods of certain rational integrals i,ii. *Annals of Mathematics*, 90, 1969.

[LF99] Paulo Lima-Filho. Algebraic families of E-infinity spectra. *Topology and its applications*, 98:217–233, 1999.

[LJ89] H. Blaine Lawson Jr. Algebraic cycles and homotoypy theory. *Ann. of Math. II*, 129:253–291, 1989.

[LJ95] H. Blaine Lawson Jr. *Spaces of Algebraic Cycles*, volume 2 of *Surveys in Differential Geometry*, pages 137–213. International Press, 1995.

[LJLFM96] H. Blaine Lawson Jr., Paulo Lima-Filho, and Marie-Louise Michelsohn. Algebraic cycles and equivariant cohomology theories. *Proc. London Math. Soc.*, 73(3):679–720, 1996.

[Pet00] Chris Peters. Lawson homology for varieties with small chow groups and the induced filtration on the griffiths groups. *Mathematische Zeitschrift*, 234(2):209–223, June 2000.

[Seg74] Graeme B. Segal. Categories and cohomology theories. *Topology*, 13:293–312, 1974.

[Seg75] Graeme B. Segal. The multiplicative group of classical cohomology. *Q. J. Math. Oxf. II*, pages 289–293, 1975.

[Tot95] Burt Totaro. Homology operations on a new infinite loop space. *Transactions of the American Mathematical Society*, 347:99–110, 1995.