Applications of Integration

6.1: More about areas

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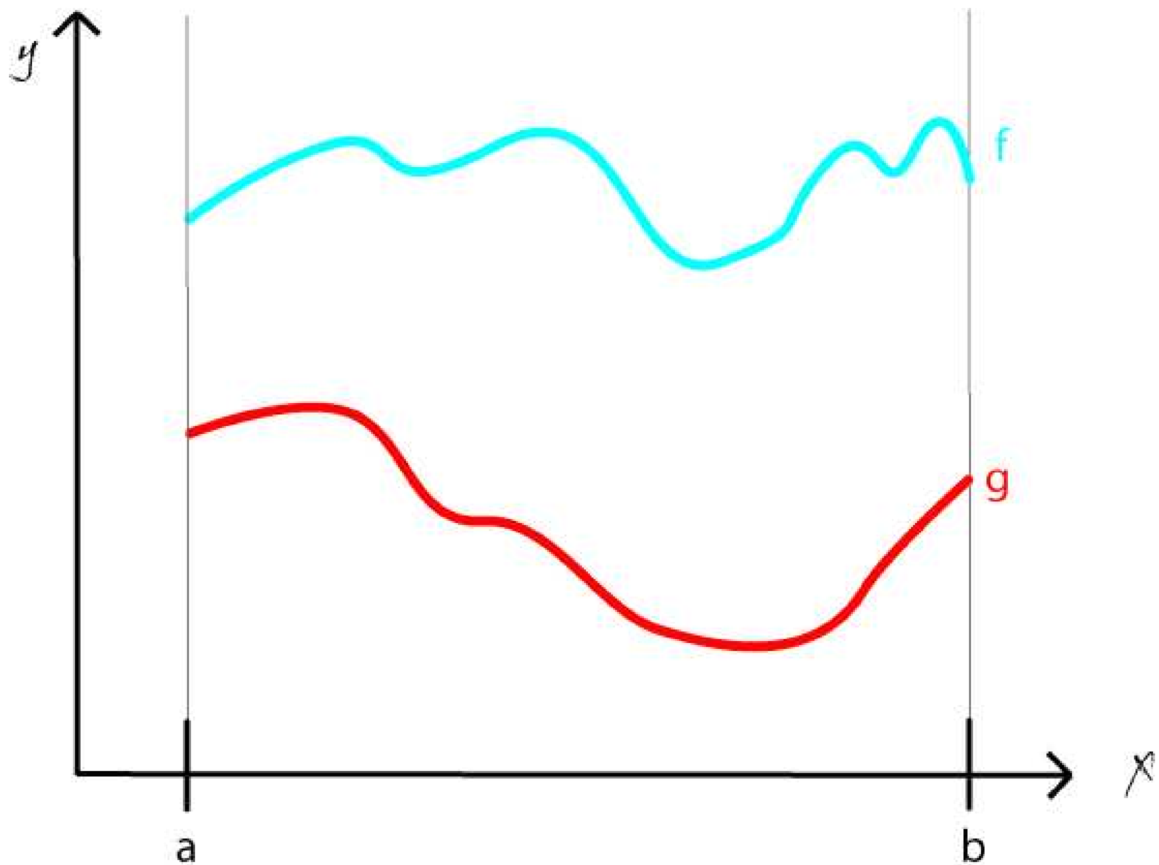
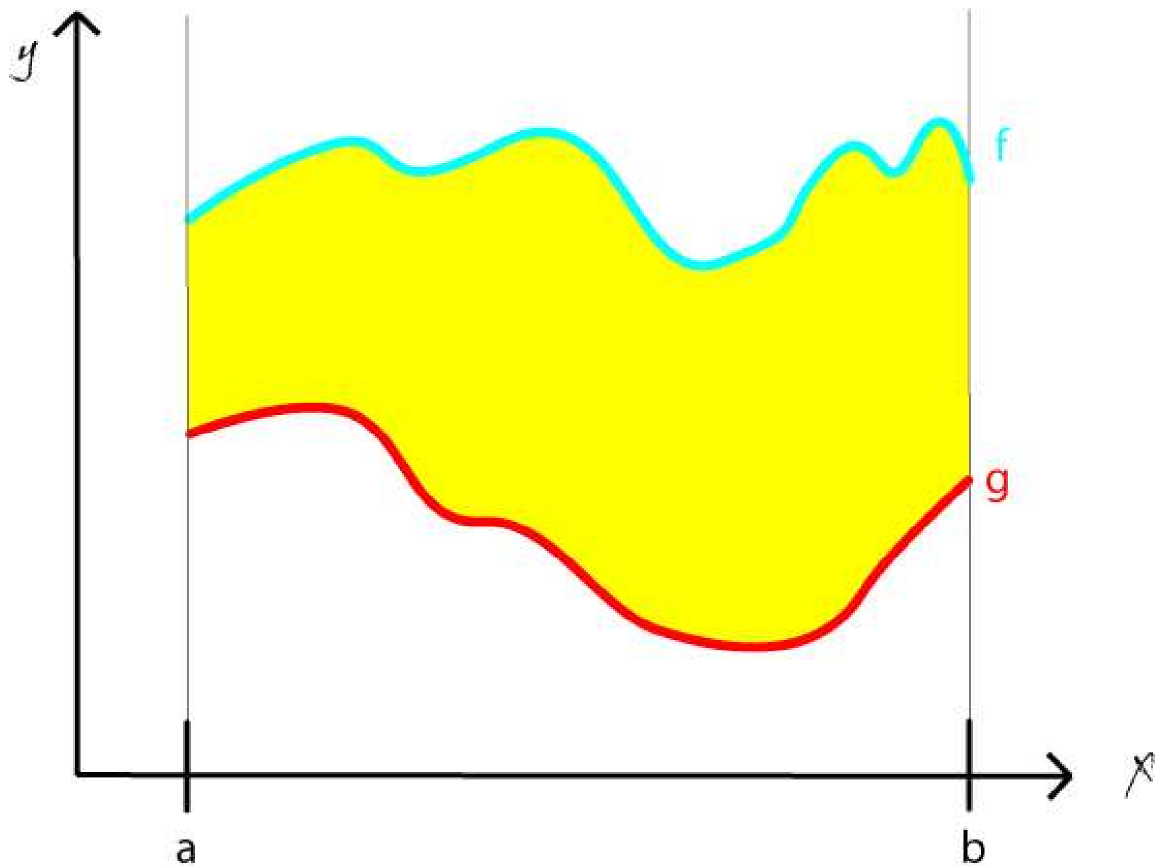
Area of a region bounded by two curves

Suppose that f and g are two continuous functions defined on an interval $[a, b]$, such that $g(x) \leq f(x)$ as in the picture:



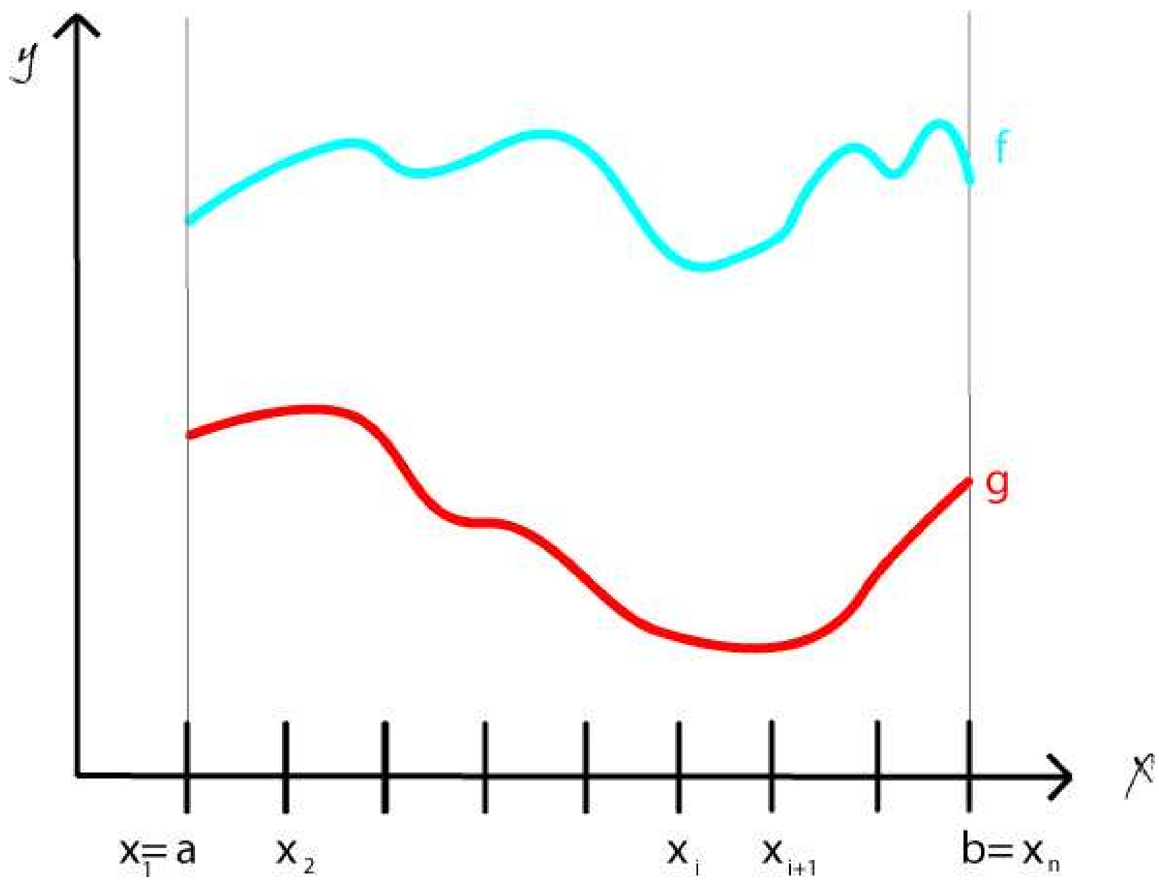
Area of a region bounded by two curves

Problem: Find the area bounded by the two curves between a and b



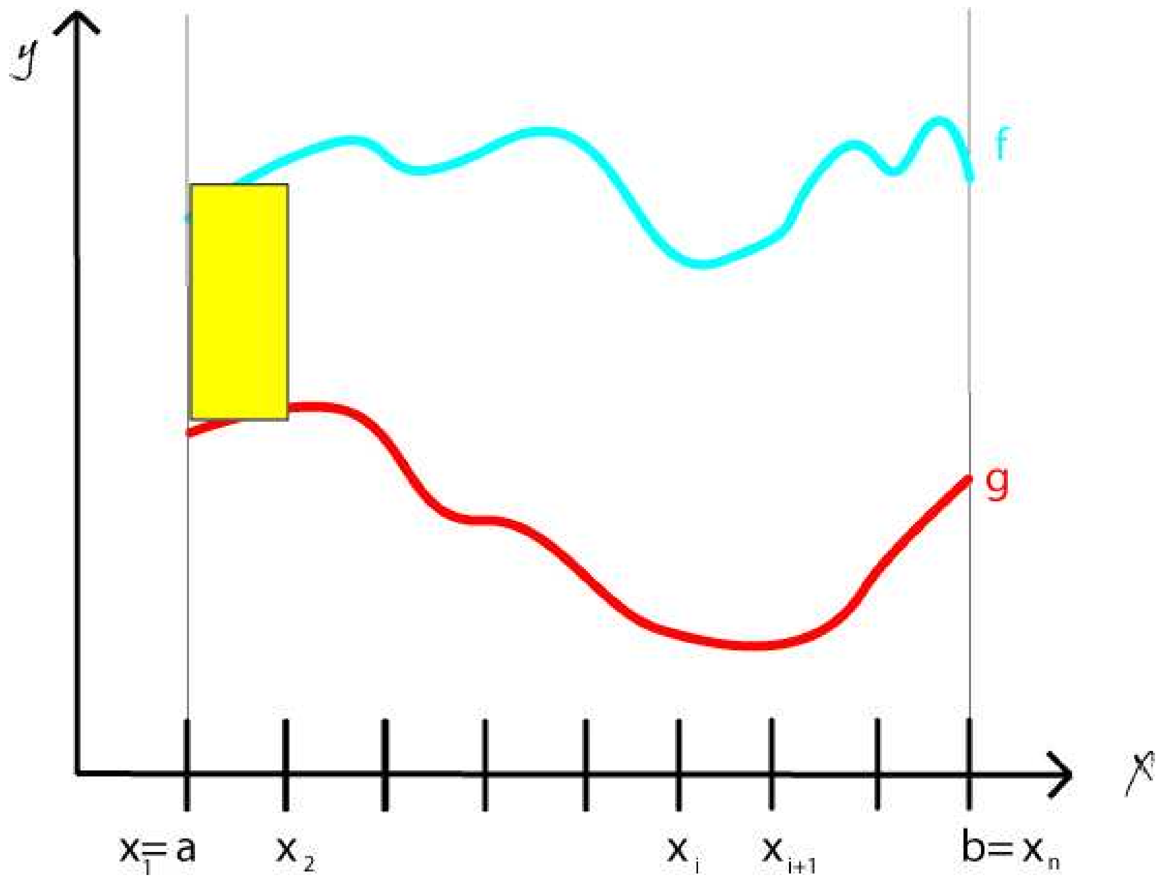
Area of a region bounded by two curves

Solution: As usual, we start by dividing the interval $[a, b]$ into subintervals determined by the endpoints $a = x_1, x_2, \ldots, x_n = b$



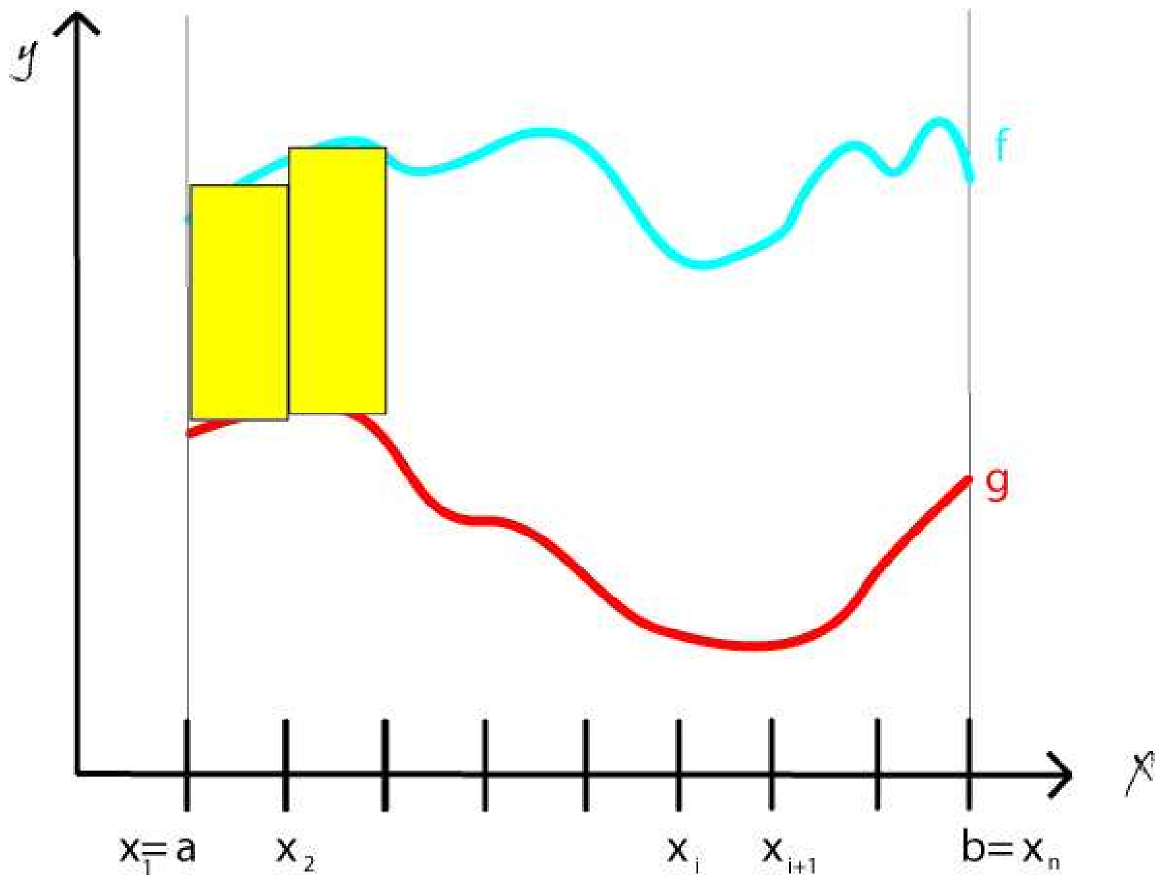
Area of a region bounded by two curves

And then we aproximate with a rectangle the area bounded by the two curves on the interval $[x_1, x_2]$



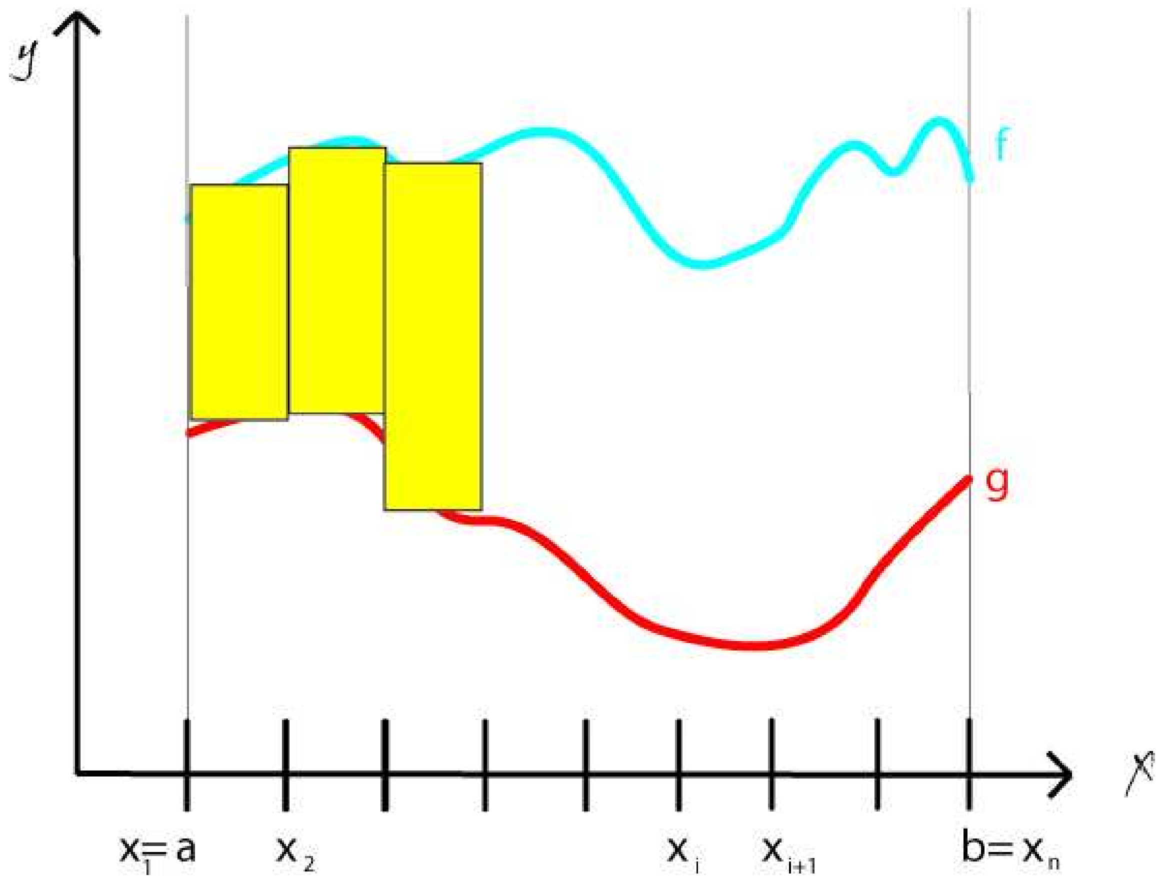
Area of a region bounded by two curves

Analogously we aproximate with a rectangle the area bounded by the two curves on the interval $[x_2, x_3]$ and we add it to the area of the previous rectangle



Area of a region bounded by two curves

And so on...



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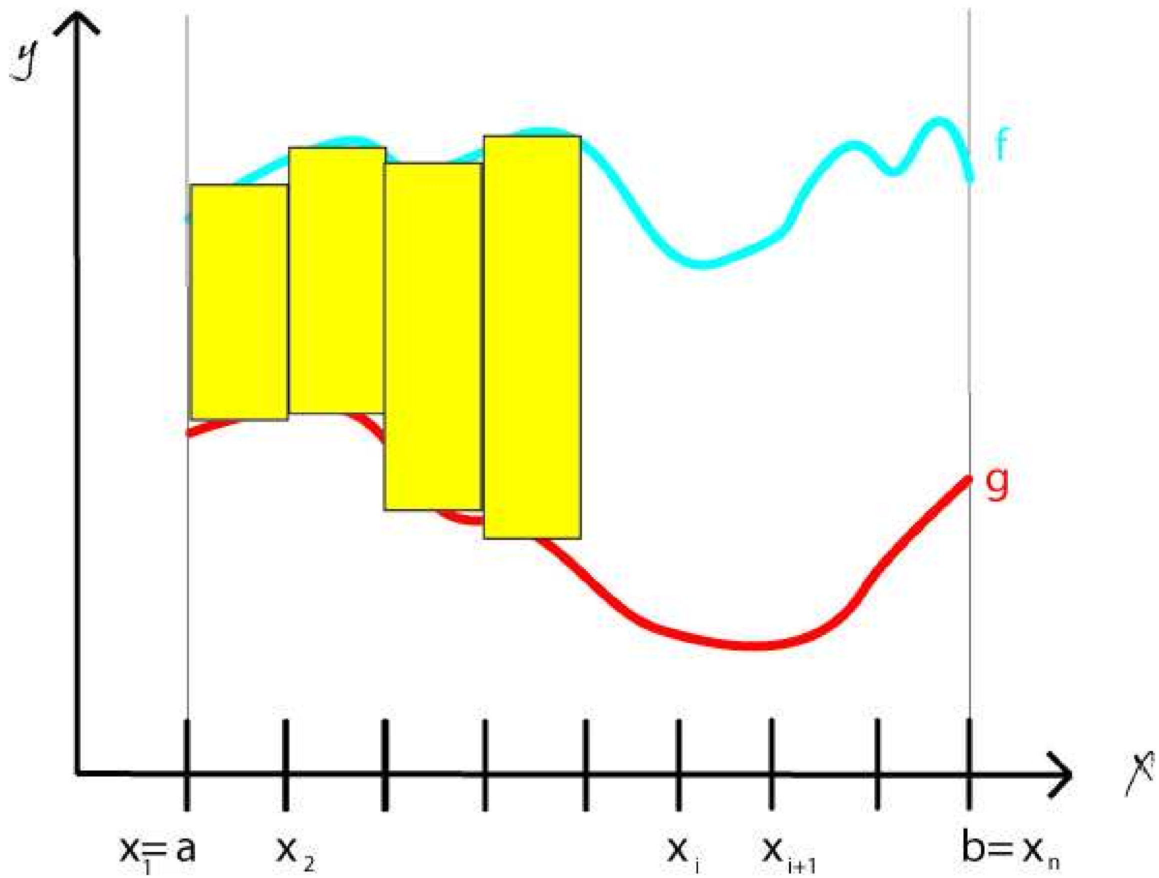

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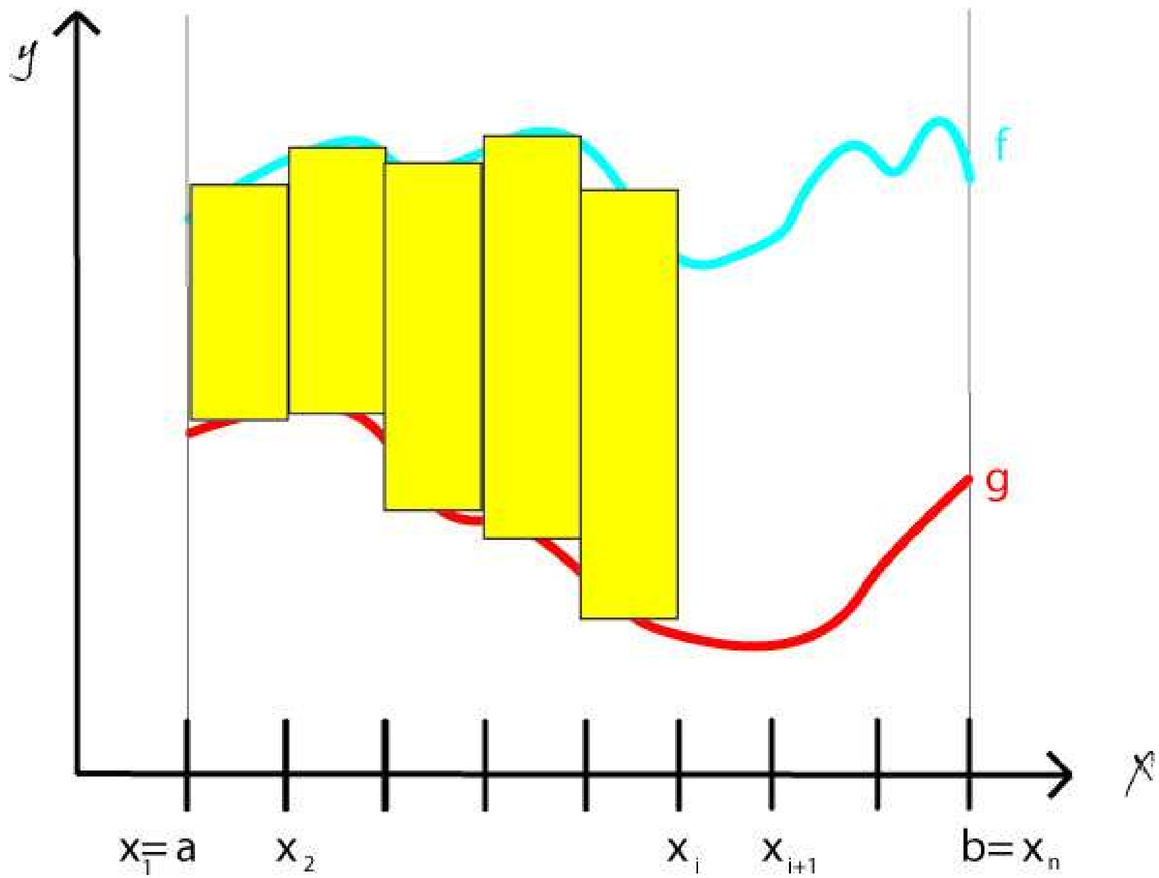

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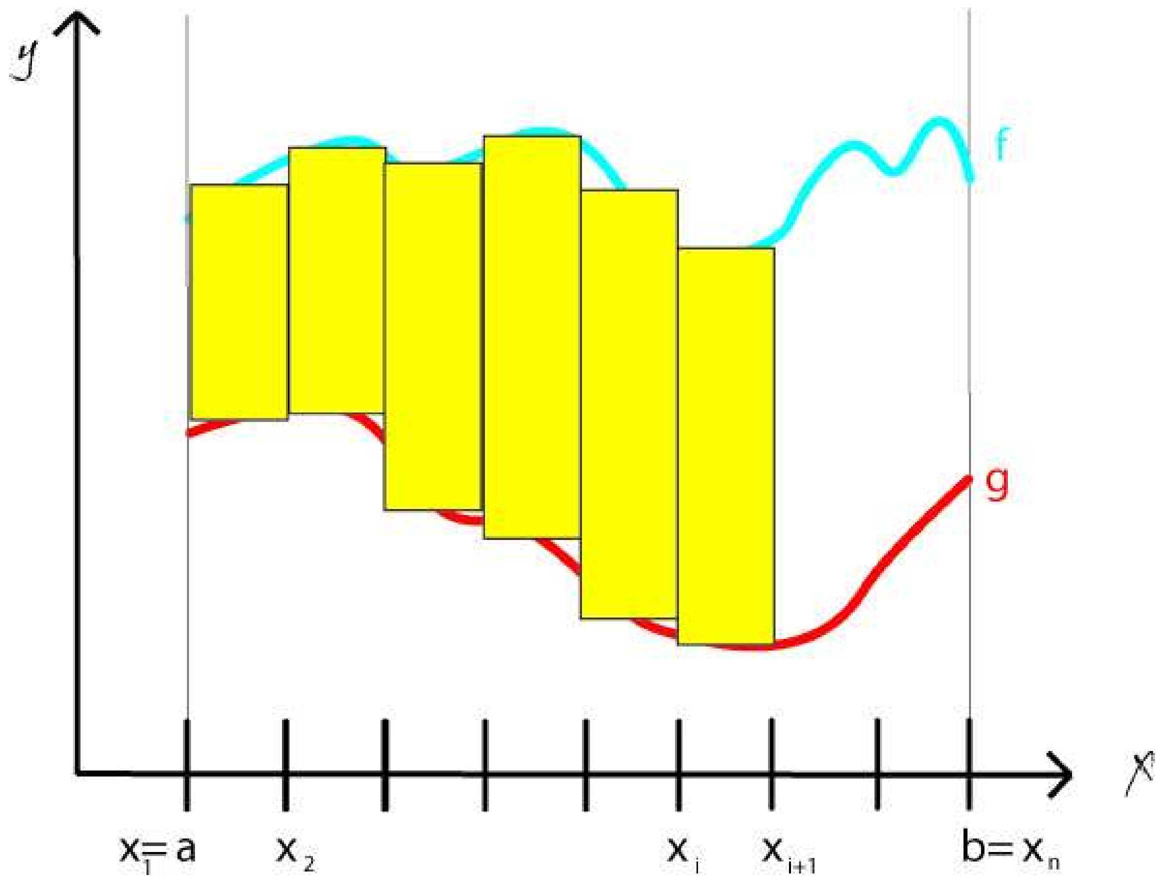

Area of a region bounded by two curves

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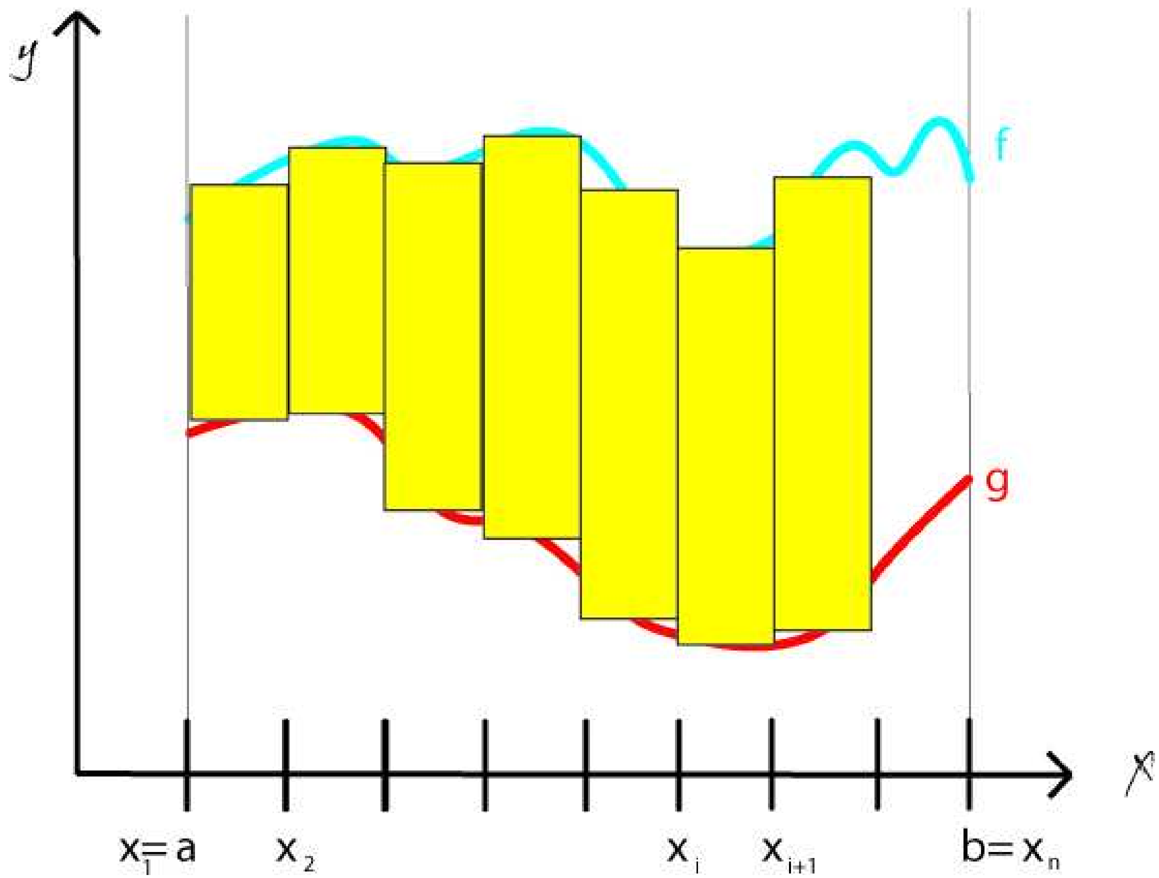
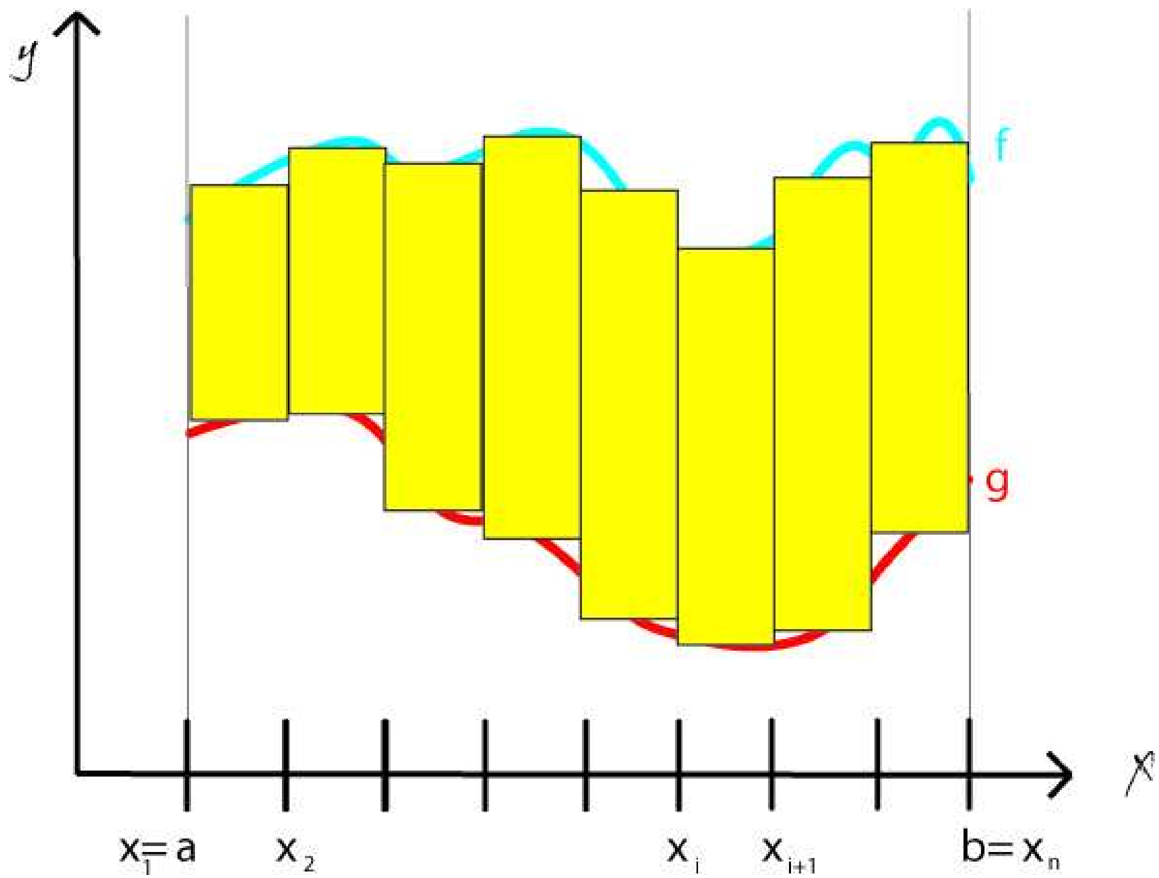


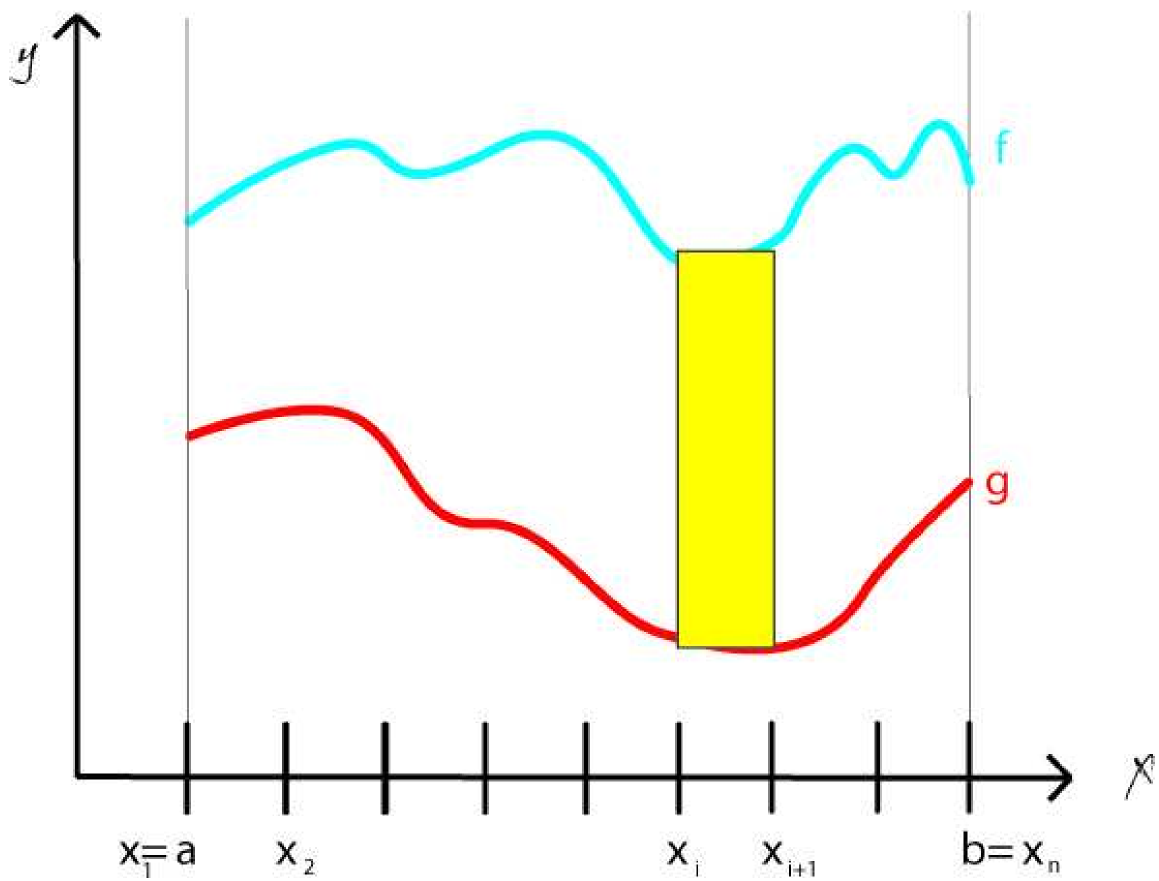
Area of a region bounded by two curves

Until we are done!



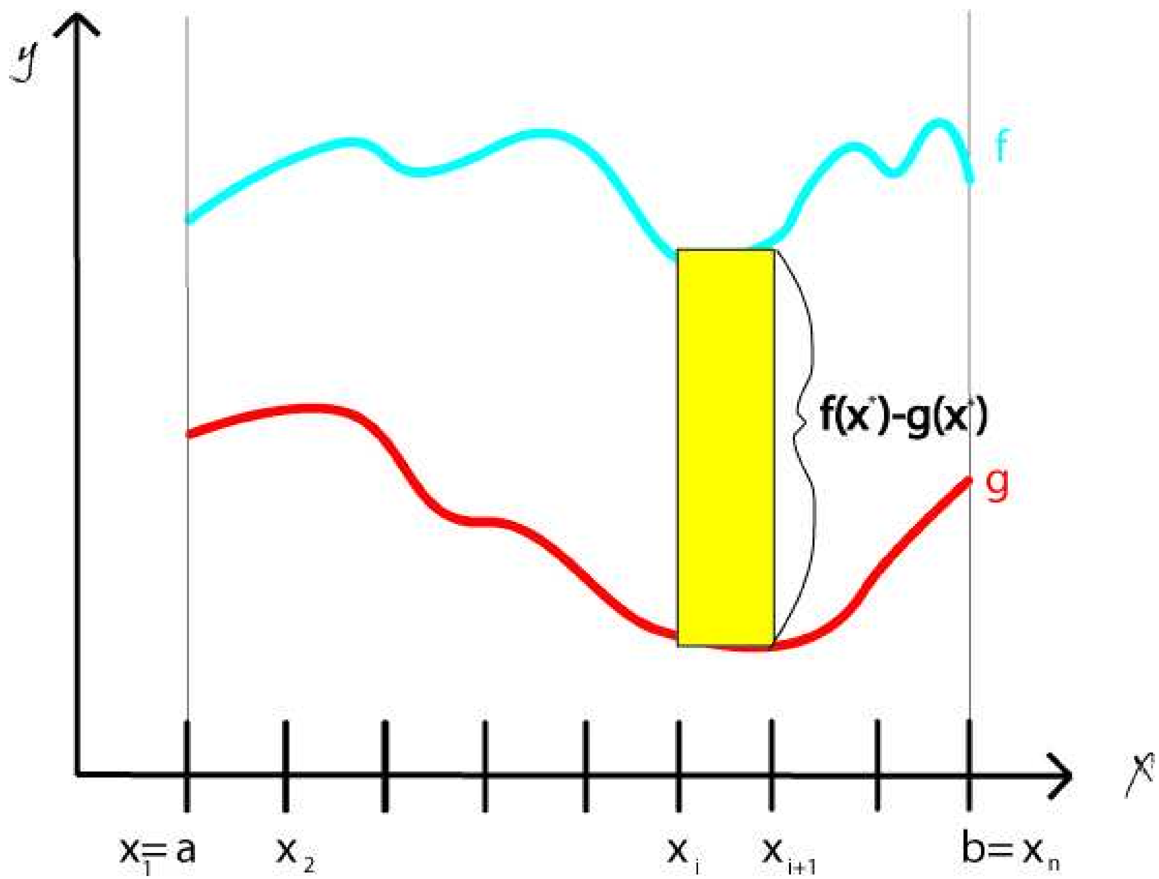
Area of a region bounded by two curves

Now, let us examine what are the formulas that describe this process. We may start by calculating the area of (any)one of the rectangles.



Area of a region bounded by two curves

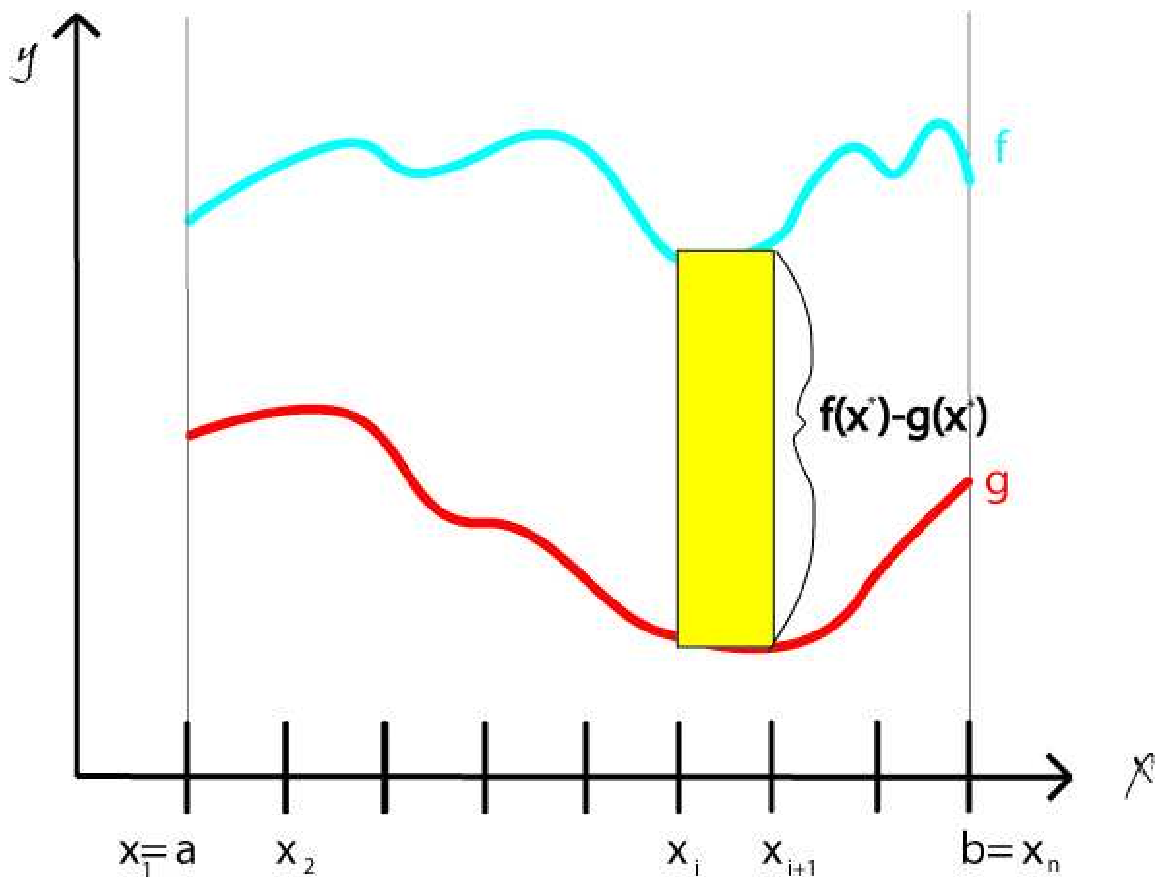
The "width" of the rectangle is $\Delta x = x_{i+1} - x_i$ and the "length" of the rectangle is $f(x_i^*) - g(x_i^*)$ where x_i^* is any point between x_i and x_{i+1}



Area of a region bounded by two curves

Therefore, the area of the rectangle is

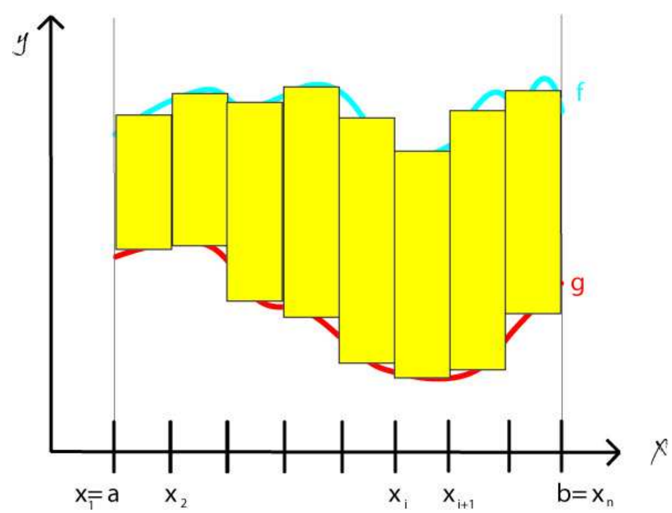
$$[f(x_i^*) - g(x_i^*)]\Delta x$$



Area of a region bounded by two curves

Now, if we add the areas of all the rectangles, the expression that we obtain for the approximation A_n is

$$A_n = \sum_{i=1}^{n} (f(x_i) - g(x_i))\Delta x$$



Area of a region bounded by two curves

Thus, the area of the region bounded by the curves is

$$\lim_{n\to\infty} A_n = \int_a^b (f(x) - g(x))dx$$

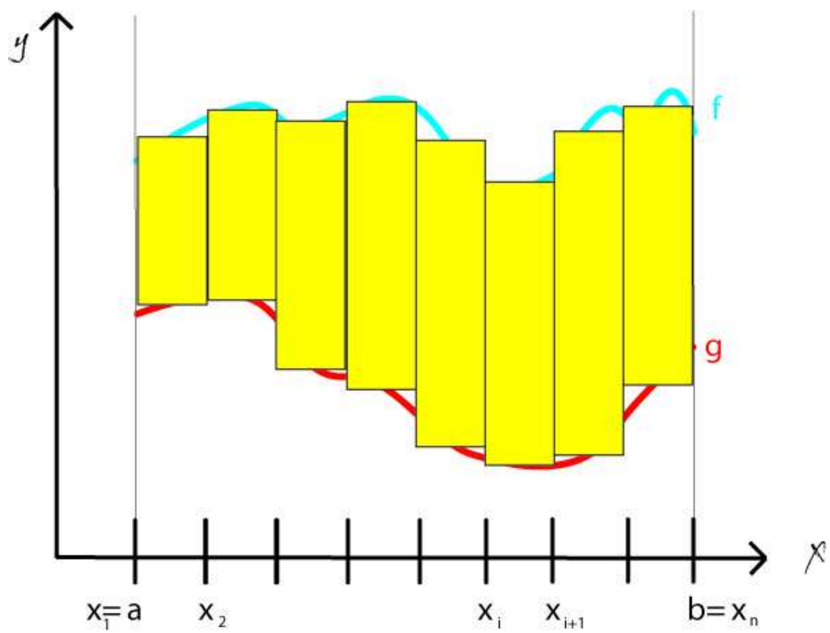


Figure 1: $n = 9$

Area of a region bounded by two curves

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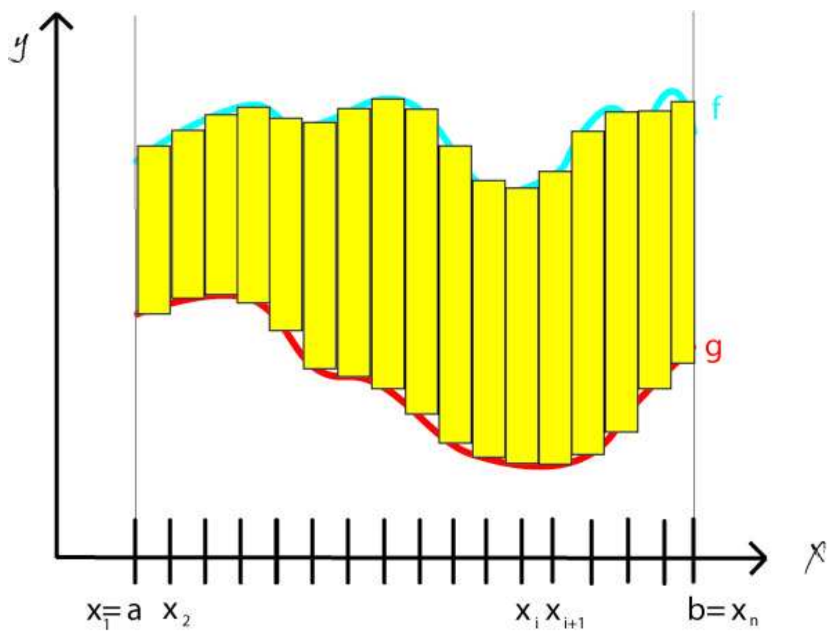

Figure 2: $n = 18$

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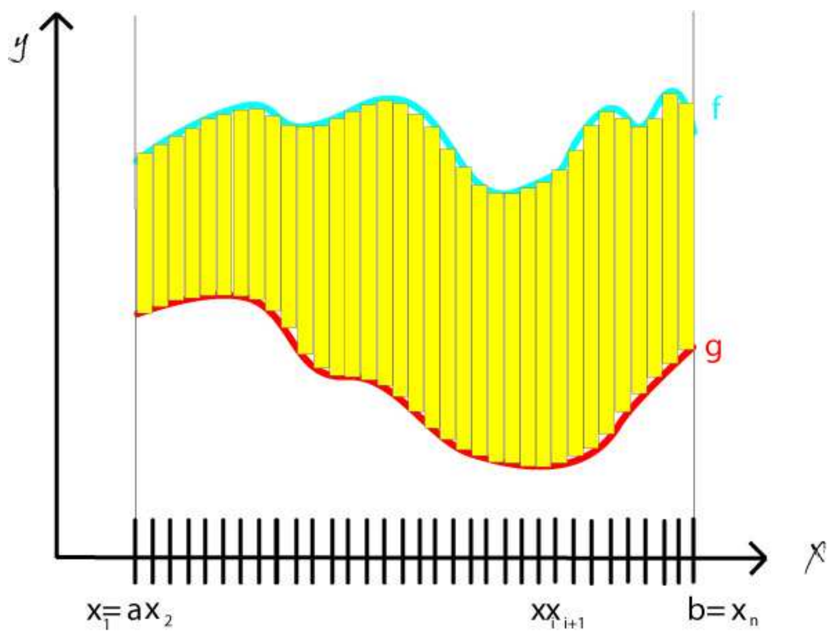

Figure 3: $n = 36$

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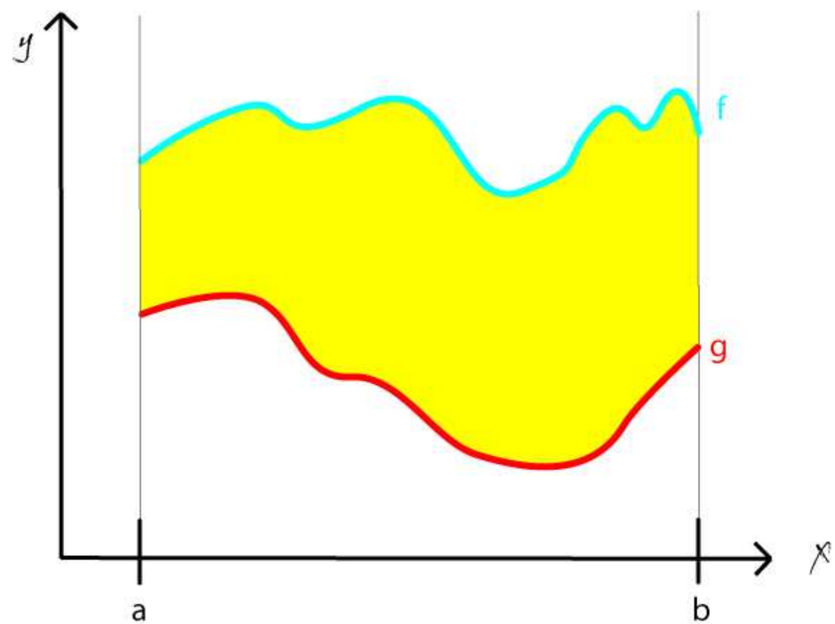
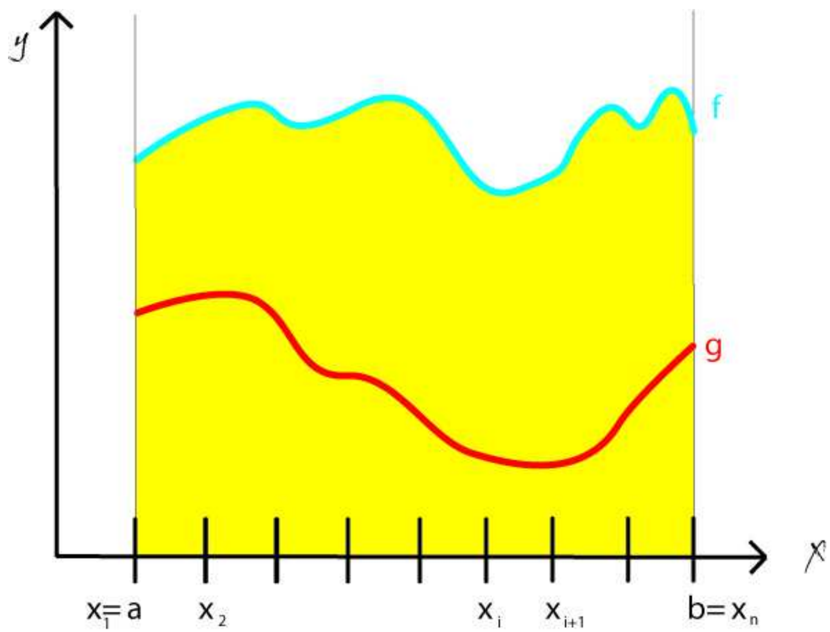


Figure 4: $\lim\limits_{n\to\infty}$

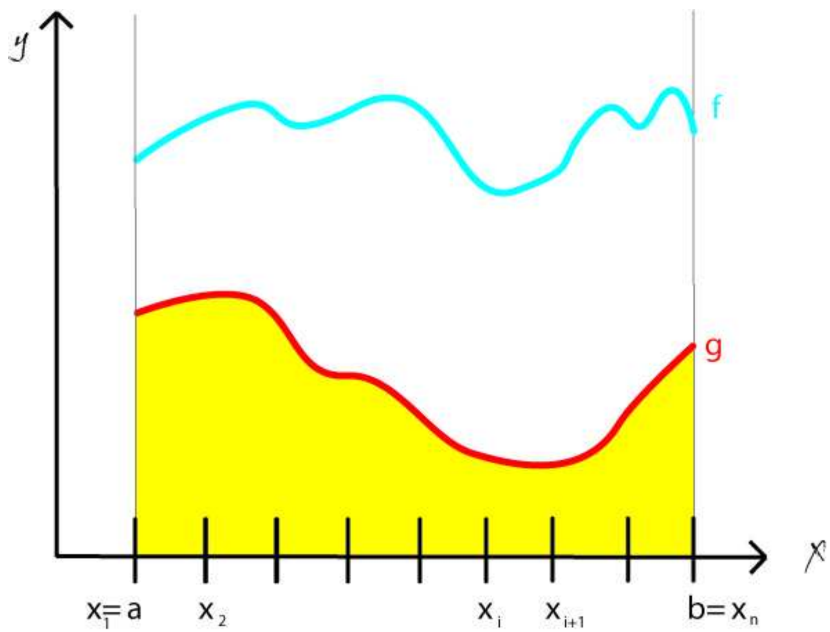
Area of a region bounded by two curves

Notice that we could also solve the problem by first calculating the area bounded by the graph of f and the x-axis



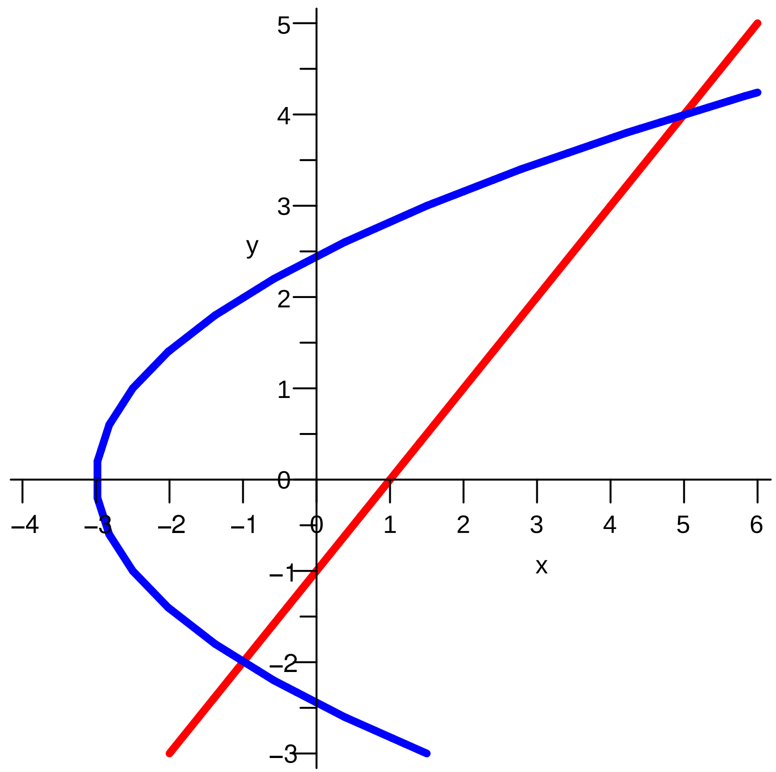
Area of a region bounded by two curves

And then we subtract the area bounded by the graph of g and the x-axis



Example 1

Find the area bounded by the curves $y = x - 1$ and $y^2 = 2x + 6$



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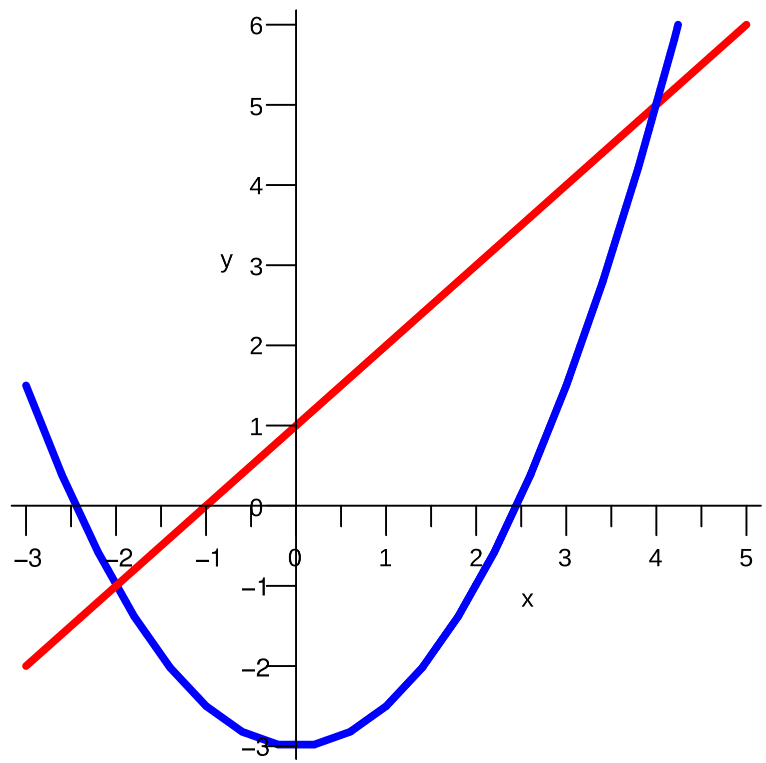
And the parabola has the equation

$$x = \frac{y^2}{2} - 3$$

Example 1

With y representing the independent variable on the horizontal axis and the vertical axis representing x, we get the following graph:



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Therefore the area bounded by the two curves is equal to the following integral:

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Challenge: Calculate this area integrating with respect to x

Example 2

Find the area of the crescent-shaped region (called a *lune*) bounded by arcs of circles with radii r and R.