MAT 127 Calculus C Fall 2002
Practice Midterm II Solutions

Name:____

I.D.:__________________________ **Section number:** _________

Please answer each question in the space provided and on the reverse side of the sheets. Show your work whenever possible. Unless otherwise indicated, **answers without justification will get little or no partial credit!** Cross out anything that grader should ignore and circle or box the final answer. You **do not need** to simplify numerical answers or write their approximate values: if the answer you got is $\sqrt{2}$ you should not replace it by 1.414. Pay close attention to the distinction between sequences of numbers and series (which are sums). The actual exam will contain 5 problems. This practice test contains more problems to give you more practice.

1. Determine whether each of the following sequences converges.

(a)

$$\left\{\frac{\sqrt{n}}{\sqrt{n}+1}\right\}_{n=1}^{\infty}$$

(b)

$$\left\{\frac{n^2+2}{n+1000}\right\}_{n=5}^{\infty}$$

(c)

$$\left\{e^{1+1/n}\right\}_{n=1}^{\infty}$$

(d)

$$\left\{\frac{\sin^2 n}{n}\right\}_{n=1}^{\infty}$$

Solution

(a)

$$\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}+1}=\lim_{n\to\infty}\frac{1}{1+1/\sqrt{n}}=1$$

— converges by the ratio law for limits.

(b)

$$\lim_{n\to\infty}\frac{n^2+2}{n+1000}=\lim_{n\to\infty}\frac{n+2/n}{1+1000/n}=\infty$$

— diverges by the ratio law for limits.

(c)

$$\lim_{n\to\infty} e^{1+1/n} = e$$

— converges since e^x is continuous (at $x = 1$).

(d)

$$\lim_{n\to\infty} \frac{\sin^2 n}{n} = 0$$

— converges by the squeeze theorem since $0 < \sin^2 n < 1$.

2. Determine whether each of the following series is convergent or divergent. Justify your answer and state which test (Integral, Comparison, p-Series, etc.) you are using. NOTE: if the series converges, you do not need to find its sum.

(a)

$$\sum_{n=1}^{\infty} \frac{10000}{1+2^n}$$

(b)

$$\sum_{n=1}^{\infty} \frac{3^{n+5}}{4^n}$$

(c)

$$\sum_{n=0}^{\infty} (-1)^n n$$

(d)

$$\sum_{n=10}^{\infty} \frac{5}{n^2 - 2n + 1}$$

(e)

$$\sum_{n=1} (-1)^n n^{-3/2}$$

(f)

$$\sum_{n=1}^{\infty} \frac{\cos(2n^2)}{n^2}$$

(g)

$$\sum_{n=5}^{\infty} \frac{1}{n \ln n}$$

Solution

(a)

$$\sum_{n=1}^{\infty} \frac{10000}{1+2^n}$$

— converges by comparison with geometric series: $a_n < 1000/2^n$.

(b)

$$\sum_{n=1}^{\infty} \frac{3^{n+5}}{4^n}$$

— converges; it is geometric series with $r = 3/4$.

(c)

$$\sum_{n=0}^{\infty} (-1)^n n$$

— diverges, since $\lim_{n\to\infty} a_n$ does not exist.

(d)

$$\sum_{n=10}^{\infty} \frac{5}{n^2 - 2n + 1}$$

— converges by the limit comparison test with the p-Series $b_n = 5/n^2$ with $p = 2$.

(e)

$$\sum_{n=1} (-1)^n n^{-3/2}$$

— converges as the difference of convergent series, $a_n = (a_n + |a_n|) - |a_n|$. These series are convergent by the comparison test with p-Series $b_n = 1/n^{3/2}$ with $p = 3/2$.

(f)

$$\sum_{n=1}^{\infty} \frac{\cos(2n^2)}{n^2}$$

— converges as the difference of convergent series, $a_n = (a_n + |a_n|) - |a_n|$. These series are convergent by the comparison test with p-Series $b_n = 1/n^2$ with $p = 2$ using $|\cos 2n^2| \leq 1$.

(g)

$$\sum_{n=5}^{\infty} \frac{1}{n \ln n}$$

— diverges by the integral test for $f(x) = 1/x \ln x$. The test is applicable since $f(x)$ is continuous, positive and decreasing on $[5, \infty)$ (because each of the functions x and

$\ln x$ is increasing there). Using the substitution $u = \ln x$ we get:

$$\int_5^t \frac{1}{x \ln x} dx = \ln \ln x|_5^t = \ln \ln t - \ln \ln 5$$

and it has no limit (goes to ∞) as $t \to \infty$.

3. Determine whether the following series is convergent or divergent.

$$\sum_{n=1} n^2 e^{-n^3}$$

Solution

Converges by the integral test for $f(x) = x^2 e^{-x^3}$. The test is applicable since $f(x)$ is continuous, positive and decreasing on $[1, \infty)$. The latter follows from the derivative test:

$$f'(x) = 2xe^{-x^3} - 3x^4 e^{-x^3} = (2 - 3x^3)xe^{-x^3} < 0$$

whenever $x \geq 1$. Using the substitution $u = x^3$ we get

$$\int_1^t x^2 e^{-x^3} dx = \frac{1}{3} \int_1^{t^3} e^{-u} du = \frac{1}{3}(e^{-1} - e^{-t^3}) \longrightarrow e^{-1}/3 \text{ as } t \to \infty.$$

4. It is given that

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Let $A = 1 + 1/2^2 + 1/3^2 + \cdots + 1/1000^2$. Estimate the difference $|\pi^2/6 - A|$. Is A greater than or less than $\pi^2/6$?

Solution

By the remainder estimate for the integral test,

$$0 < \pi^2/6 - A = \pi^2/6 - s_{1000} = R_{1000} \leq \int_n^\infty \frac{dx}{x^2}.$$

We have

$$\int_n^\infty \frac{dx}{x^2} = \lim_{t \to \infty} \int_n^t \frac{dx}{x^2} = \lim_{t \to \infty} (-1/t + 1/n) = 1/n.$$

Therefore $|\pi^2/6 - A| = R_{1000} \leq 1/1000 = 0.001$ and A is less than $\pi^2/6$.

5. Express the number $5.404040404040\ldots$ as a ratio of integers.

Solution

$$5.404040\ldots = 5 + 40/100 + 40/100^2 + 40/100^3 + \cdots = 5 + 40\sum_{n=1}^{\infty}\frac{1}{100^n}$$

$$= 5 + \frac{40}{1 - 1/100} = 5 + 40/99 = 535/99.$$

6. A sequence is defined recursively as

$$a_1 = 1,\ \ a_{n+1} = a_n/2n.$$

Determine whether the series

$$\sum_{n=1}^{\infty} a_n$$

converges.

Solution

We have for all $n > 1$ that $a_{n+1} < a_n/2$, so

$$a_{n+1} < a_n/2 < a_{n-1}/2^2 < \cdots \leq a_1/2^n = 1/2^n.$$

Comparison with the geometric series with $r = 1/2$ shows that the series converges.

7. A hard rubber ball has the property that after being dropped from a height h onto a hard surface, it bounces back up to a height rh, where $r < 1$. If the ball is dropped from an initial height of H meters, and continues to bounce indefinitely, what will be the total distance that the ball travels?

Solution

The distance traveled by the ball is

$$H+rH+rH+r^2H+r^2H+\cdots = H+\sum_{n=1}^{\infty}2Hr^n = H+\frac{2Hr}{1-r} = \frac{H+Hr}{1-r}.$$