MAT 127 Calculus C Fall 2002
Midterm II Solutions

1. (a) (10 points) Express the number 3.212121... as a ratio of integers.

Solution

$$3.212121\ldots = 3 + \frac{21}{100} + \frac{21}{100^2} + \cdots = 3 + \sum_{n=1}^{\infty} \frac{21}{100^n}$$
$$= 3 + \frac{21}{100} \cdot \frac{1}{1 - 1/100} = 3 + \frac{21}{99} = \frac{318}{99},$$

where we have used that the sum is geometric series with $a = 21/100$ and $r = 1/100$. The answer is represented as the ratio of integers (which is a fraction).

(b) (10 points) For what value of q does

$$\sum_{n=1}^{\infty} q^n$$

equal 2?

Solution

This is geometric series with $a = r = q$. Assuming that the series converges, we find that its sum is $q/(1-q)$. Thus we have an equation

$$\frac{q}{1-q} = 2, \text{ that is } q = 2 - 2q, \text{ or } 3q = 2,$$

so that $q = 2/3$.

2. Determine whether each of the following sequences converges or diverges. If it converges, find the limit. In either case, justify your answer.

Solutions

(a) (4 points)

$$a_n = \frac{5n^2 + 3}{6n^3 + 2}$$

It converges and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5/n + 3/n^3}{6 + 2/n^3} = \frac{0}{6} = 0.$$

(b) (3 points)

$$a_n = \cos \pi n$$

It diverges since $a_n = (-1)^n$, i.e., it is 1 for even n and -1 for odd n.

(c) (6 points)

$$a_n = \ln(n^2+1) - \ln(n^2)$$

It converges and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{n^2+1}{n^2}\right) = \lim_{n\to\infty} \ln\left(1+\frac{1}{n^2}\right) = \ln 1 = 0.$$

(d) (6 points)

$$a_n = \frac{n}{\ln n}$$

It diverges by the L'Hospital rule, since $x' = 1$, $(\ln x)' = 1/x$ and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\frac{1}{n}} = \lim_{n\to\infty} n = \infty.$$

(e) (6 points)

$$a_n = \frac{\sin n}{\sqrt{n}}$$

It converges by the squeeze theorem: $|\sin n| \leq 1$ and $\lim_{n\to\infty} 1/\sqrt{n} = 0$, so that the sequence converges to 0.

3. (30 points) Determine whether each of the following series is convergent or divergent. Justify your answer and state which test (Integral, Comparison, p-Series, etc.) you are using. NOTE: if the series converges, you do not need to find its sum.

Solutions

(a) (4 points)

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

It is convergent by the comparison test with convergent p-Series $b_n = 1/n^2$ with $p = 2$, because

$$\frac{1}{n^2+1} < \frac{1}{n^2}$$

for all n.

(b) (6 points)

$$\sum_{n=1}^{\infty} \frac{6}{n^2-3n+10}$$

It is convergent by the limit comparison test with convergent p-Series $b_n = 1/n^2$ with $p = 2$, because

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{6n^2}{n^2-3n+10} = \lim_{n\to\infty} \frac{6}{1-3/n+10/n^2} = 6 > 0$$

(c) (3 points)

$$\sum_{n=1}^{\infty}(-1)^{n^2}$$

It is divergent by the divergence test since $\lim_{n\to\infty} a_n$ does not exist: $a_n = 1$ for even n and $a_n = -1$ for odd n.

(d) (4 points)

$$\sum_{n=10}^{\infty}\frac{\sin^2 n}{3^n+5n}$$

It is convergent by comparison test with convergent geometric series $b_n = 1/3^n$ with $r = 1/3$, because

$$0 < \frac{\sin^2 n}{3^n+5n} < \frac{1}{3^n}$$

for all $n = 1, 2, \dots$

(e) (7 points)

$$\sum_{n=1}^{\infty}(2ne^{-n^2}+3e^{-n})$$

It is convergent: the second series is the geometric series with $r = 1/e < 1$, whereas the first series is subject to the integral test with $f(x) = 2xe^{-x^2}$. Indeed, $f(x)$ is continuous, positive and decreasing on $[1,\infty)$ since

$$f'(x) = 2e^{-x^2} - 4x^2e^{-x^2} = 2(1-2x^2)e^{-x^2} < 0$$

whenever $x \geq 1$. Using the substitution $u = x^2$, we get

$$\int_1^{\infty} f(x)dx = \lim_{t\to\infty}\int_1^t 2xe^{-x^2}dx = \lim_{t\to\infty}\int_1^{t^2} e^{-u}du$$
$$= \lim_{t\to\infty}\left(e^{-1}-e^{-t^2}\right) = e^{-1}$$

— the improper integral is convergent.

NOTE One can also use the integral test applied to the function $2xe^{-x^2}+3e^{-x}$, though it is an extra work since we already know that geometric series in question converges.

(f) (6 points)

$$\sum_{n=1}^{\infty}\frac{\ln n}{n}$$

It is divergent by the comparison test with harmonic series $b_n = 1/n$ (harmonic series is divergent), because

$$\frac{\ln n}{n} > \frac{1}{n}$$

whenever $n \geq 3$.
NOTE One can also use the integral test applied to the function $f(x) = \ln x/x$ and use the sustitution $u = \ln x$, though it would require more work.

4. (15 points) Determine whether the series

$$\sum_{n=3}^{\infty} \frac{2}{n(n-1)}$$

is convergent or divergent. If it is convergent, find its sum.
Solution
It is convergent telescopic series. Indeed, by partial fractions,

$$\frac{2}{n(n-1)} = \frac{2}{n-1} - \frac{2}{n}$$

so that

$$\sum_{n=3}^{\infty} a_n = \sum_{n=3}^{\infty} \left(\frac{2}{n-1} - \frac{2}{n} \right)$$
$$= \left(\frac{2}{2} - \frac{2}{3} \right) + \left(\frac{2}{3} - \frac{2}{4} \right) + \left(\frac{2}{4} - \frac{2}{5} \right) + \cdots = 1$$

since $s_n = 1 - 2/n$ converges to 1.

5. (10 points) It is given that

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Give an estimate for the error $|s_{100} - \pi^2/8|$. Is s_{100} greater or less than $\pi^2/8$?
Solution
By the remainder estimate for the integral test,

$$R_n \leq \int_n^{\infty} f(x)dx, \text{ where } f(x) = \frac{1}{(2x+1)^2}$$

and $|\pi^2/8 - s_{100}| \leq R_{100}$. We have, doing the standard integral,

$$\int_n^t \frac{dx}{(2x+1)^2} = \frac{1}{2(2n+1)} - \frac{1}{2(2t+1)} \longrightarrow \frac{1}{2(2n+1)}$$

as $t \to \infty$. Thus $R_{100} \leq 1/2(201) = 1/402$ and

$$0 < \pi^2 - s_{100} \leq 1/402,$$

since obviuosly s_{100} is less than $s = \pi^2/8$ — the sum of the series.