MAT 127 Calculus C Fall 2002
Solutions To Midterm I

Name:____

I.D.:_________________________ **Section number:** _________

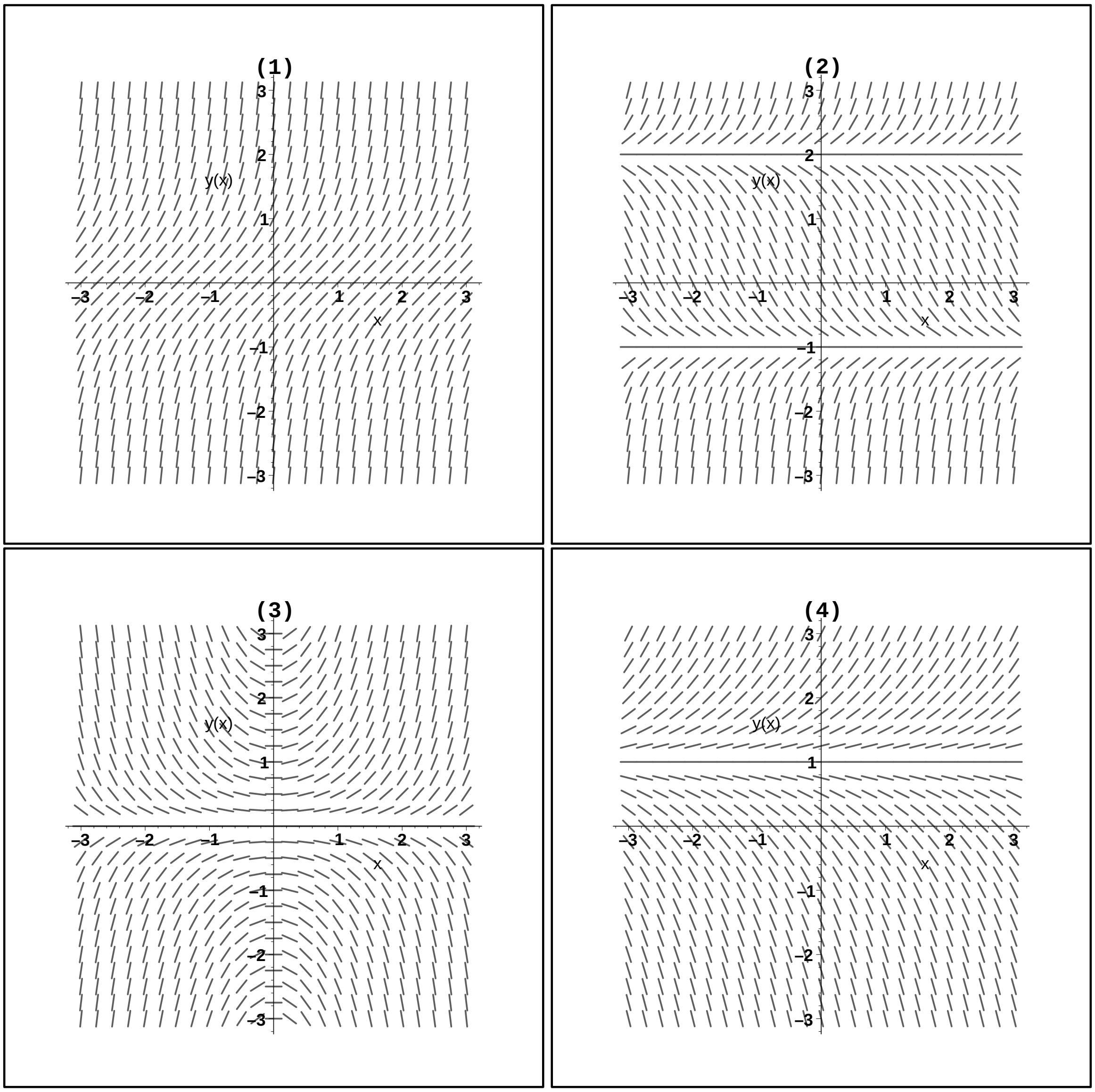
1	2	3	4	5	TOTAL
20 points	20 points	20 points	20 points	20 points	100 points

No books, notes, or calculators!

Please answer each question in the space provided and on the reverse side of the sheets. Show your work whenever possible. Unless otherwise indicated, **answers without justification will get little or no partial credit!** Cross out anything that grader should ignore and circle or box the final answer. You **do not need** to simplify numerical answers or write their approximate values: if the answer you got is $\sqrt{2}$ you should not replace it by 1.414. Double-check your answers and remember to include units in word problems! Check that your exam has all 5 problems.

1. Shown below are 4 differential equations and 4 direction fields.

(a) $y' = y - 1$ (b) $y' = xy$ (c) $y' = (y+1)(y-2)$ (d) $y' = y^2 + 1$



(i) (10 points) Match the equations and the corresponding direction fields (no justification required).

(ii) (10 points) In each case, find all equilibrium solutions.

Solution (both parts, (i) and (ii))

First, look at the equilibrium solutions: equation (a) has $y = 1$, (b) has $y = 0$, (c) has $y = -1$ and $y = 2$, (d) has none. Checking against direction fields we easily see that (a) goes with (4), (b) goes with (3) (in this case the y-axis also has zero slope), (c) — with (2), and (d) — with (1).

2. (i) (15 points) For what values of A and B does the function $y(x) = Ax \sin x + B \cos x$ satisfy the differential equation

$$y'' + y = 2\cos x.$$

(ii) (5 points) Repeat part (i) for the function

$$y(x) = Ax \sin x + B \cos x + C \sin x$$

Solution

(i) We successively compute

$$y' = A \sin x + Ax \cos x - B \sin x,$$

$$y'' = A\cos x + A\cos x - Ax\sin x - B\cos x = 2A\cos x - Ax\sin x - B\cos x.$$

This gives

$$y'' + y = 2A\cos x.$$

Thus if differential equation is to be satisfied, we should have $A = 1$. The fact that we get no equation for B merely means that B can be any number! (This, of course, means, that the function $B\cos x$ is a solution to the homogeneous equation $y'' + y = 0$). So $\boxed{A = 1, B = \text{any real number}}$

(ii) Observe that the function $C \sin x$ is also a solution to the equation $y'' + y = 0$. So $\boxed{A = 1, B = \text{any real number}, C = \text{any real number}}$

3. Consider the differential equation $xyy' = \ln x$.

(i) (15 points) Find all solutions $y(x)$.

(ii) (5 points) Find the solution to the initial value problem $y(1) = 3$.

Solution

(i) We have the equation for differentials

$$ydy = \frac{\ln x\, dx}{x},$$

i.e.,

$$\int ydy = \int \frac{\ln x\, dx}{x}.$$

The last integral is readily computed using the substitution $u = \ln x$, $du = dx/x$, and we get

$$\frac{y^2}{2} = \frac{(\ln x)^2}{2} + \text{const}, \text{ or } y^2 = (\ln x)^2 + C.$$

So $\boxed{y = \pm\left((\ln x)^2 + C\right)^{1/2}}$

(ii) If $y(1) = 3$, then, using $\ln 1 = 0$, we get $C = 9$. So the solution to the initial value problem is $\boxed{y = \pm\left((\ln x)^2 + 9\right)^{1/2}}$

4. After 6 days a sample of the radioactive material lost 75% of its mass.

(i) (10 points) Find the half-life of the material.

(ii) (10 points) After how long only 12.5% of the material's mass will remain?

Solution

(i) If m was the initial mass of the material and r is the relative decay rate, then we have

$$me^{-6r} = \frac{m}{4},$$

where t is measured in days. Solving for r we get $e^{-6r} = 1/4$, or $e^{6r} = 4$, so that $6r = \ln 4 = 2\ln 2$, i.e.,

$$r = \ln 2/3.$$

The half-life T of the material and its relative decay rate r are related by the formula $rT = \ln 2$, which follows from the equation

$$me^{-rT} = \frac{m}{2}, \text{ or } e^{rT} = 2.$$

Using the formula for r, we get $\boxed{T = 3 \text{ days}}$

(ii) Now we want to find time T_1 such that

$$me^{-rT_1} = \frac{m}{8}.$$

We get $e^{rT_1} = 8$, or $rT_1 = \ln 8 = 3\ln 2$. Using the formula for r, we get $\boxed{T_1 = 9 \text{ days}}$

5. A tank is filled with 5 liters of salt water with a salt concentration 50 grams per liter. A salty water with concentration of salt 10 grams per liter flows into the tank at the rate 1 liter per hour. The liquid in tank is well-mixed and the excess leaves out so that the volume is 5 liters at all times.

(i) (5 points) Write a differential equation for $S(t)$, the amount of salt in the tank at time t (t is measured in hours, $S(t)$ is measured in grams). What initial condition does $S(t)$ satisfy?

(ii) (10 points) At what time does the concentration of salt equals to 20 grams per liter?

(iii) (5 points) What happens to $S(t)$ as t becomes very large?

Solution

(i) As usual in the mixing problems, we write

$$\frac{dS}{dt} = \text{rate in} - \text{rate out}$$

The rate at which salt enters the tank is $10g/L$ (the concentration of the solution entering) times $1L/h$ (the rate at which fluid enters). The rate at which it leaves is $S(t)/5$ (the concentration of salt in the tank) times $1L/h$ (the rate at which fluid leaves; it is the same as the entering rate since total volume stays the same). So the differential equation is

$$\boxed{\frac{dS}{dt} = 10 - \frac{S}{5}}$$

The initial condition is $\boxed{S(0) = 250 \text{ grams}}$ — the amount of liquid, 5 liters, times the initial concentration of salt, $50g/L$.

(ii) We have to solve the initial value problem found in the first part.

$$\int \frac{dS}{10 - S/5} = \int dt,$$

or $(-5)\ln|10 - S/5| = t + C$. Solving for S and renaming the constant gives $S(t) = 50 + Ae^{-t/5}$. Setting $S(0) = 250$ we find that $A = 200$, i.e., that

$$S(t) = 50 + 200e^{-t/5} \text{ grams.}$$

Now for concentration to equal $20g/L$, the amount of salt in the tank must be $20g/L$ times $5L$, the volume of the tank. So we need to find t such that $S(t) = 20 \cdot 5 = 100$ grams. So we set

$$50 + 200e^{-t/5} = 100$$

and find that

$$e^{-t/5} = \frac{50}{200} = \frac{1}{4}, \text{ or } e^{t/5} = 4.$$

Thus $t/5 = \ln 4 = 2\ln 2$, so that $\boxed{t = 10\ln 2}$

(iii) As $t \to \infty$, the term $200e^{-t/5}$ becomes arbitrarily small (it decays exponentially), so that

$$\boxed{\lim_{t\to\infty} S(t) = 50 \text{ grams}}$$