MAT 551 Real Analysis II Fall 2003
Midterm

1. Let $\Omega \subset \mathbb{C}$ be a domain and $a \in \Omega$. Prove that the function

$$\frac{1}{z-a}$$

defines a distribution on $C_0^\infty(\Omega)$ and that in $C_0^\infty(\Omega)'$,

$$\frac{\partial}{\partial \bar{z}}\left(\frac{1}{z-a}\right) = \pi\delta_a.$$

(Hint: for the second statement, use Stokes' theorem)

2. Let $l_\infty(\mathbb{R}) = \{a \in l_\infty : a_n \in \mathbb{R} \text{ for all } n\}$. Prove that there exists a linear functional l on $l_\infty(\mathbb{R})$ such that

$$\liminf a_n \leq l(a) \leq \limsup a_n$$

for all $a = \{a_n\}_{n=1}^\infty \in l_\infty(\mathbb{R})$.

3. Prove that every bounded set in a Hilbert space is weakly compact.

4. Let $T \in \mathcal{L}(\mathcal{H})$ and $c > 0$. Prove that the operator $T^*T + cI$ is self-adjoint and invertible.

5. (a) Let $\mathcal{H} = L^2(0,1)$ and let $K(x,t)$ be continuous on $[0,1] \times [0,1]$. Prove that the integral Volterra operator

$$(Tf)(x) = \int_0^x K(x,t)f(t)dt$$

is compact and has no non-zero eigenvalues.

(b) Find the spectral radius and the norm of the operator

$$(Tf)(x) = \int_0^x f(t)dt.$$

6. Let $\mathcal{H} = l_2$ and let S be the shift operator,

$$S(a_1, a_2, \dots) = (0, a_1, a_2, \dots).$$

Suppose that the operator D,

$$D(a_2, a_2, \dots) = (\lambda_1 a_1, \lambda_2 a_2, \dots),$$

is compact. Prove that the spectral radius of SD is zero.

7. Let $T \in \mathcal{L}(\mathcal{H})$ be compact operator.

(a) Prove that T and T^* have the same singular values.

(b) Prove that T and UTV, where U and V are unitary, have the same singular values.

8. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Let P be an orthogonal projection onto the subspace $\mathbb{C}e_1$, and let

$$f_n = (n^2+n)^{-\frac{1}{2}} \left(\sum_{k=1}^{n} e_k - n e_{n+1} \right).$$

(a) Show that $\|P\|_1 = 1$.
(b) Show that $\{f_n\}_{n=1}^{\infty}$ is an orhtonormal basis for $\mathcal{H}$.
(c) Show that $\sum_n \|Pf_n\| = \infty$.

9. Let $\{a_{ij}\}_{i,j=1}^{\infty}$ satisfy

$$\sum_{i,j=1}^{\infty} |a_{ij}|^2 < \infty.$$

(a) Prove that

$$T(v_1, v_2, \dots) = (w_1, w_2, \dots), \quad w_j = \sum_{k=1}^{\infty} a_{jk} v_k,$$

defines a bounded operator on $\mathcal{H} = l_2$.
(b) Prove that T is Hilbert-Schmidt.
(c) Find $\|T\|_2$.