MAT 551 Real Analysis II Fall 2003
Final Exam

1. Let (X, μ) be a measure space. Prove that the Banach space $L^\infty(X, d\mu)$ is either finite-dimensional or not separable.

2. (a) Let $f \in L^1(\mathbb{R}, dx)$ and $f_\varepsilon = \varepsilon^{-1} f(\frac{x}{\varepsilon})$. Prove that in the weak topology of $\mathcal{S}'(\mathbb{R})$

$$\lim_{\varepsilon \to 0} f_\varepsilon = c\delta_0, \quad \text{where} \quad c = \int_{-\infty}^{\infty} f(x)dx.$$

(b) Verify that

$$\lim_{\varepsilon \to 0} \frac{e^{-\pi x^2/\varepsilon}}{\sqrt{\varepsilon}} = \delta(x).$$

3. (a) Prove that the function $f(x) = \exp(ie^x)$ defines a distribution $T_f \in \mathcal{S}'(\mathbb{R})$, but the distributional derivative T_f' does not correspond to function $f'(x)$.

(b) Find the general solution of the equation $x^n f(x) = 0$ in $\mathcal{S}'(\mathbb{R})$.

4. (a) Find the Fourier transform of the distribution

$$\mathcal{P}\frac{1}{x}.$$

(b) Prove that the Laplace equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = T$$

has no nonzero solutions in $\mathcal{S}'(\mathbb{R}^2)$.

5. (a) Show that the multiplication by x operator in $L^2(0,1)$ is not compact.

(b) Prove that the operator

$$T(x_1, x_2, \dots) = (a_1 x_1, a_2 x_2, \dots)$$

in l_2, where $\{a_n\}$ is a bounded sequence, is compact if and only if

$$\lim_{n \to \infty} a_n = 0.$$

6. Let $a(x) \in L^\infty(\mathbb{R})$ and let T be a muliplication by $a(x)$ operator in $L^(\mathbb{R})$. Find the spectrum of T.

7. Prove that in the complex Hilbert space $\mathcal{H}$ with the inner product (,) the following identities hold:

(a)

$$(x,y)=\frac{1}{N}\sum_{n=1}^{N}\|x+e^{2\pi in/N}y\|^2e^{2\pi in/N},\ N\geq 3.$$

(b)

$$(x,y)=\frac{1}{2\pi}\int_0^{2\pi}\|x+e^{i\theta}y\|^2e^{i\theta}d\theta.$$

8. (Ergodic theorem of von Neumann) Let U be a unitary operator in $\mathcal{H}$ and let P be a projection operator onto $\ker(U-\mathrm{I})$. Prove that

$$s-\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}U^n=P.$$

9. Let A be a multiplication by x^2 operator in $\mathcal{H}=L^2(-1,1)$.

(a) Prove that A has no cyclic vectors in $\mathcal{H}$.

(b) Prove that $\mathcal{H}$ can be decomposed into the direct sum of two subspaces $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2$ which are invariant for A and such that $\mathcal{H}_i$ has a cyclic vector for $A_i=A|_{\mathcal{H}_i}$, $i=1,2$.

10. (Extra Credit) Let A be a trace class integral operator on $L^2(0,1)$ with a kernel $K(x,y)$ continuous on $[0,1]\times[0,1]$. Prove that

$$\mathrm{tr}A=\int_0^1 K(x,x)dx.$$